The local h-polynomial of a triangulation of the simplex

Conference in Memory of Prof. Charalambides

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Introduction

A polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x]$$

with nonnegative coefficients is said to be

- symmetric, with center of symmetry $n/2$, if $a_i = a_{n-i}$ for $0 \leq i \leq n$
- unimodal if

$$a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$$

for some $0 \leq k \leq n$

- γ-positive, with center of symmetry $n/2$, if

$$f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$$

for some $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor} \geq 0$

- real-rooted if $f(x) \equiv 0$, or all complex roots of $f(x)$ are real.

Example

$$f(x) = 1 + 26x + 66x^2 + 26x^3 + x^4$$

is symmetric and unimodal, γ-positive, since

$$f(x) = (1+x)^4 + 22x(1+x)^2 + 16x^2,$$

and real-rooted.

Note

- real-rootedness $\Rightarrow$ unimodality

- $\left\{\begin{array}{l}\text{symmetry and}\\ \text{real-rootedness}\end{array}\right. \Rightarrow$

  $\Rightarrow \gamma$-positivity

  $\Rightarrow \left\{\begin{array}{l}\text{symmetry and}\\ \text{unimodality.}\end{array}\right.$

Example

We let

$[n] = \{1, 2, \ldots, n\}$

$\mathfrak{S}_n$ = group of permutations of $[n]$

$\mathcal{D}_n$ = set of permutations $w \in \mathfrak{S}_n$ without fixed points

and for $w \in \mathfrak{S}_n$

$$\mathrm{des}(w) = \#\{i \in [n-1] : w(i) > w(i+1)\}$$

$$\mathrm{exc}(w) = \#\{i \in [n-1] : w(i) > i\}$$

be the number of descents and excedances of w, respectively.

(α) The n^{th} Eulerian polynomial

$$A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{des(w)} = \sum_{w \in \mathfrak{S}_n} x^{exc(w)}$$

is

- symmetric, with center of symmetry $(n-1)/2$
- γ-positive (Foata-Schützenberger 1970)
- real-rooted (Frobenius 1910).

$$A_n(x) = \begin{cases} 1, & n=1 \\ 1+x, & n=2 \\ 1+4x+x^2, & n=3 \\ 1+11x+11x^2+x^3, & n=4 \\ 1+26x+66x^2+26x^3+x^4, & n=5 \end{cases}$$

$$= \begin{cases} 1, & n=1 \\ 1+x, & n=2 \\ (1+x)^2+2x, & n=3 \\ (1+x)^3+8x(1+x), & n=4 \\ (1+x)^4+22x(1+x)^2+16x^2, & n=5. \end{cases}$$

(b) The n^{th} derangement polynomial

$$d_n(x) = \sum_{w \in D_n} x^{exc(w)}$$

is

- symmetric, with center of symmetry $n/2$, and unimodal (Brenti 1990, Stembridge 1992)
- γ-positive (several authors)
- real-rooted (Zhang 1995, Haglund-Zhang 2019, Brändén-Solus 2021).

$$d_n(x) = \begin{cases} 0, & n=1 \\ x, & n=2 \\ x+x^2, & n=3 \\ x+7x^2+x^3, & n=4 \\ x+21x^2+21x^3+x^4, & n=5 \end{cases}$$

$$= \begin{cases} 0, & n=1 \\ x, & n=2 \\ x(1+x) & n=3 \\ x(1+x)^2+5x^2, & n=4 \\ x(1+x)^3+18x(1+x), & n=5. \end{cases}$$

How can one generalize $A_n(x)$ and $d_n(x)$?

Face enumeration of simplicial complexes

Let

Δ = simplicial complex of dimension $n-1$

$f_k(\Delta)$ = # k-dimensional faces of Δ.

Definition. The f, h-polynomials of Δ are defined as

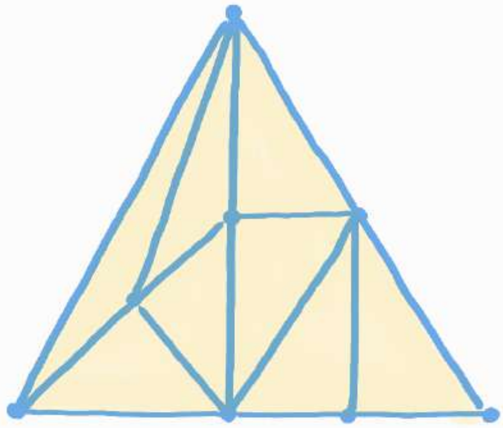
$$f(\Delta, x) = \sum_{k=0}^{n} f_{k-1}(\Delta)\, x^k$$

$$h(\Delta, x) = \sum_{k=0}^{n} f_{k-1}(\Delta)\, x^k (1-x)^{n-k}$$

$$= (1-x)^n f\left(\Delta, \frac{x}{1-x}\right).$$

Note. $h(\Delta, 1) = f_{n-1}(\Delta)$.

Example.



$n = 3$

Δ

$$f_0(\Delta) = 8, \quad f_1(\Delta) = 15, \quad f_2(\Delta) = 8$$

- $f(\Delta, x) = 1 + 8x + 15x^2 + 8x^3$
- $h(\Delta, x) = (1-x)^3 + 8x(1-x)^2 + 15x^2(1-x) + 8x^3$

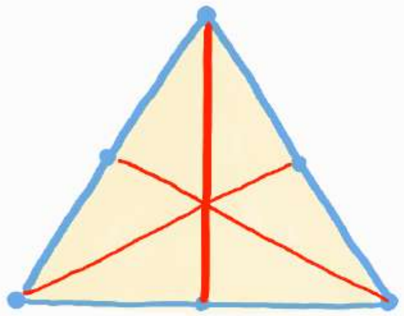
$$= 1 + 5x + 2x^2.$$

Example. We let

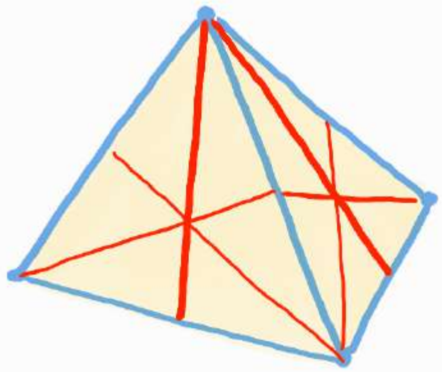
V = n-element set

2^V = abstract simplex on the vertex set V

Δ = first barycentric subdivision of 2^V.



$n=3$



$n=4$

Then, $h(\Delta, x) = A_n(x)$.

We let

V = n-element set

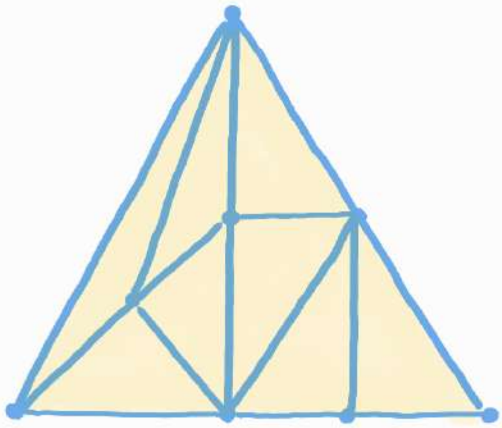
Γ = triangulation of 2^V

Γ_F = restriction of Γ on $F \in 2^V$.

Definition (Stanley 1992). The local h-polynomial of Γ (with respect to V) is defined as

$$\ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{n-|F|} h(\Gamma_F, x).$$

Example.



Γ

- $$\ell(\Gamma_V, x) = (1+5x+2x^2) - (1+2x) - (1+x) - 1 + 1 + 1 + 1 - 1$$
$$= 2x + 2x^2.$$

Theorem (Stanley 1992). For every triangulation Δ' of a pure simplicial complex Δ

$$h(\Delta', x) = \sum_{F \in \Delta} \ell_F(\Delta'_F, x)\, h(\mathrm{link}_\Delta(F), x).$$

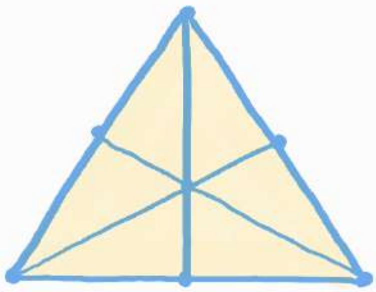
Theorem (Stanley 1992). The polynomial $\ell_V(\Gamma, x)$

- is symmetric, with center of symmetry $n/2$, for every triangulation Γ of the simplex 2^V.
- has nonnegative coefficients for every triangulation Γ of the simplex 2^V.
- is unimodal for every regular triangulation Γ of the simplex 2^V.

Conjecture (A, 2012). The polynomial $\ell(\Gamma_V, x)$ is γ-positive for every flag triangulation Γ of 2^V.

Example. If

- V = n-element set
- Δ = first barycentric subdivision of 2^V



then

- $$\ell_V(\Gamma, x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k(x)$$

$$= \sum_{w \in D_n} x^{\mathrm{exc}(w)} = d_n(x).$$

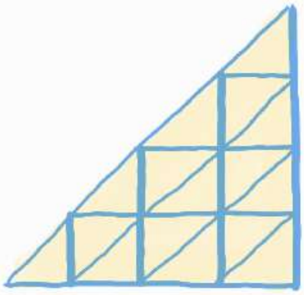
In particular,

- $$\ell_V(\Gamma, 1) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k(1)$$

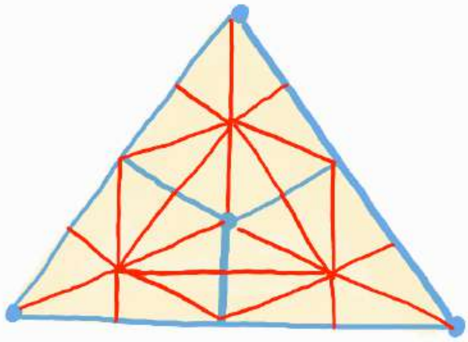
$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k!$$

$$= \# D_n = d_n(1).$$

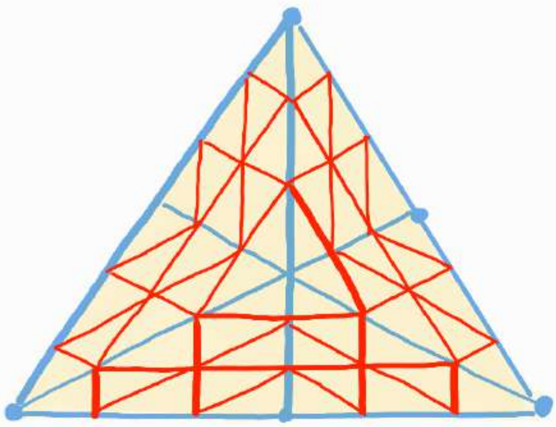
Combinatorial interpretations of $\ell_V(\Gamma, x)$ have been found for



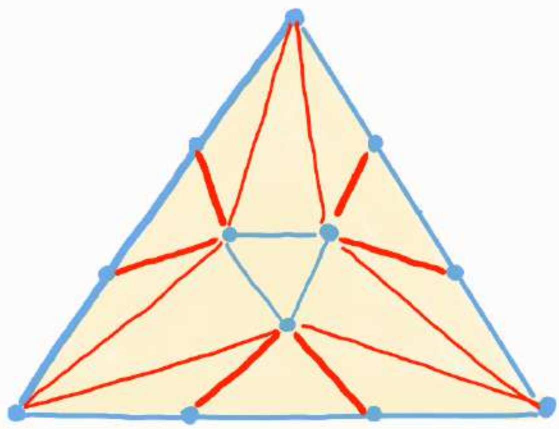
r-fold edgewise subdivision



interval triangulations



r-colored barycentric subdivisions



antiprism triangulations

For instance:

Proposition (A-Kubitzke-Brunink, 2022) For the antiprism triangulation Γ_n of the $(n-1)$-dimensional simplex 2^V

$$[x^k]\, \ell_V(\Gamma, x) = \binom{n}{k} \# \{ w \in \mathcal{D}_n : \mathrm{Exc}(w) = [k] \}$$

for $0 \le k \le n$, where

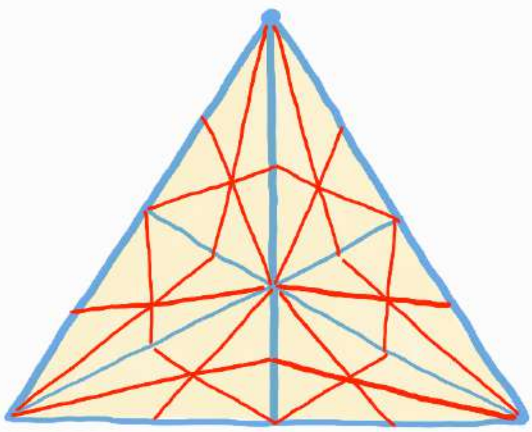
$$\mathrm{Exc}(w) = \{ i \in [n-1] : w(i) > i \}$$

for $w \in \mathfrak{S}_n$.

Notation. Let $sd^{(k)}(\Delta)$ be the k^{th} barycentric subdivision of a complex Δ.

Question (A, 2016) Find a combinatorial interpretation of

$$\ell_V(sd^{(2)}(2^V), x)$$



Is this polynomial real-rooted?

$$\ell_V(sd^{(2)}(2^V), x) =$$

$$= \begin{cases} 0, & n=1 \\ 3x, & n=2 \\ 13x+13x^2, & n=3 \\ 75x+303x^2+75x^3, & n=4. \end{cases}$$

Note. Setting $sd^{(2)}(2^V) = sd_n^{(2)}$ for $\# V = n$,

- $\ell_V(sd_n^{(2)}, x) =$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} h(sd_k^{(2)}, x)$$

- $\ell_V(sd_n^{(2)}, 1) =$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k!)^2$$

$= \# \, (u,v) \in \mathfrak{S}_n \times \mathfrak{S}_n :$ u and v have no common fixed point.

For $n = 0, 1, 2, \ldots$ these are the numbers

1, 0, 3, 26, 453, 11844, 439975, ...

and are studied in

Ch. A. Charalambides, Enumerative Combinatorics, Chapman & Hall/CRC, 2002, p. 187, Exercise 13(a).

let

$$D_{n,k} = \# \{ w \in D_n : \operatorname{exc}(w) = k \}$$

$$d_{n,k,j}(x) = \sum_{\substack{w \in \mathfrak{S}_{n+1} : \operatorname{Fix}(w) \subseteq [n+1-k] \\ w^{-1}(1) = j+1}} x^{\operatorname{exc}(w)} .$$

Theorem (A, 2024+). (a)

$$\ell_V(\mathrm{sd}^{(2)}(2^V), x) = \sum_{k+j \leq n} \binom{n}{k} D_{n-k,j}\, d_{n,k,j}(x)$$

where $n = |V|$. Equivalently, the coefficient of x^i in $\ell_V(\mathrm{sd}^{(2)}(2^V), x)$ equals the number of

$$(u,v) \in \mathfrak{S}_n \times \mathfrak{S}_{n+1} : \begin{cases} \mathrm{Fix}(v) \subseteq [n - \mathrm{fix}(u) + 1] \\ v^{-1}(1) = \mathrm{exc}(u) \\ \mathrm{exc}(v) = i. \end{cases}$$

(b) $\ell_V(\mathrm{sd}^{(k)}(2^V), x)$ is real-rooted for all n, k.

Thank you for your attention

Ευχαριστώ για την προσοχή σας

Спасибо за внимание!