Composition schemes:
q-enumerations and phase transitions in Gibbs models


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International Conference on Combinatorial Methods and Probability Models
A conference in memory of professor Charalambos A. Charalambides
Athens, October 5-6, 2024

Article: `https://lipn.fr/~banderier/Papers/Gibbs.pdf`, published in LIPICS Proceedings of AofA 2024.

Charalambos was the organizer of the "5th International Conference on Lattice Path Combinatorics and Discrete Distributions" (Athens, Greece, June 5-7, 2002).
⇝ I went there with my PhD advisor, Philippe Flajolet, and I gave a talk on "Why Delannoy numbers?" (cf. previous talk by Christian!)

Anecdote: French people are considered to have a tradition of long lunches/dinners, but I remember it was however unusual for us that the conference lunch was at ∼ 3pm30! 🙂

Since, I also got in charge of this conference, and the next Lattice Path Conference (Canada, summer 2026) will be dedicated to the memory of Professor Charalambides.


https://lipn.univ-paris13.fr/~banderier/LPC/

List of past Lattice Path Conferences   Recent event: Lattice Path'2021

History of the Lattice Path Conference

"Reminiscing over: a short historical view on the series of conferences", by its founder Sri Gopal Mohanty

[This text is based on the text written by Gopal for the Siena conference in 2010, a version of which was published in Fundamenta Informaticae 117 (2012). For this web version, it was slightly modified/updated with photos and links added.]

The two first books dedicated to lattice paths, by Narayana and Mohanty in 1979.

Just after almost simultaneous publications in 1979 of two books, "Lattice Path Combinatorics with Statistical Applications", by Tadepalli Venkata Narayana and "Lattice Path Counting and Applications", by me, I realized that there was a substantial growing interest in lattice path combinatorics, with applications in computer science, statistics and applied probability. I also realized that the distribution of researchers was world wide. In order to increase the awareness of the subject, my intention to organize a conference to bring eminent and young researchers together and to promote interaction between the theory group and those involved in applications resulted in the first Conference on Lattice Path Combinatorics and Applications that was held at McMaster University, Canada in 1984. Incidentally, I have been at McMaster University since 1964 and the University was highly supportive of my initiative to organize the conference. Its success prompted quite a few to voice an *encore* for it. In the mean time the publication of two books, "Combinatorial Enumeration" by Ian Goulden and David Jackson in 1983 and "Enumerative Combinatorics - Vol 1" by Richard Stanley in 1986 encouraged me to organize another conference.

Lattice Path'1990, McMaster

B. L. S. Prakasa Rao, Endre Csáki, István Vincze, Gopal Mohanty at the Lattice Path Conference, in 1994 at Delhi.

Lattice Path'1994, Delhi

Lattice Path'1998, Wien

The second conference was held again at McMaster University in 1990, although some of the enthusiasts wished it to be held earlier. Both conferences had international participation and triggered so much interest that participants showed their willingness to organize next events. Thus, subsequent conferences were called International and were held at University of Delhi in 1994, University of Vienna in 1998, University of Athens in 2002 and East Tennessee State University in 2007, University of Siena in 2010, Cal Poly Pomona university in 2015. The main local organizers of these events were Kanwar Sen in India, Walter Böhm and Christian Krattenthaler in Austria, Charalambos A. Charalambides in Greece, Anant Godbole in USA, Renzo Pinzani and Simone Rinaldi in Italy, Alan Krinik in USA in 2015. The international nature is also reflected by regular participations from Australia, Austria, Bangladesh, Canada, China, France, Germany, Greece, India, Italy, Japan, Kazakhstan, South Korea, South Africa, Sweden, Taiwan, UK, and USA.

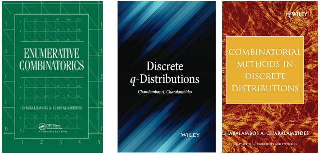
Lattice Path'2007, Johnson City

Lattice Path'2010, Siena

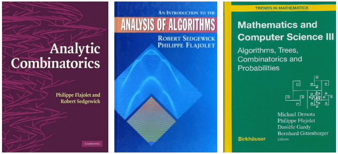
Charalambos Charalambides and Philippe Flajolet

To my eyes, Charalambos was a Greek “Philippe Flajolet”:
Both were enjoying food, cigarettes, beer, and also kindly serving as a mentor for many students & older researchers in combinatorics and probability theory!

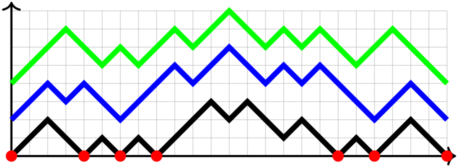
Charalambos Charalambides (1945-2024)



Philippe Flajolet (1948-2011)



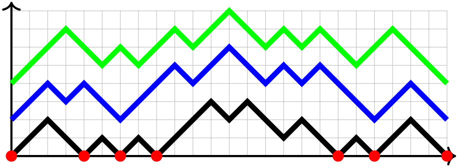
Many common keywords: Catalan/Stirling/Eulerian numbers, partitions, arrangements, permutations, generating functions, walks, Markov chains, distributions, orthogonal polynomials, urns, q-analogues. . .



Number of x-axis contacts for m-watermelons, counted with weight $q^{\#\text{contacts}}$

$\rightsquigarrow$ mean has completely different asymptotics for different values of q!
What could be the corresponding limit laws?
Is there a phase transition at $q = 1$? ($q < 1$ repulsive, $q > 1$ attractive?)

An open problem of Christian Krattenthaler



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No context-free grammar, too hard model to be solved?

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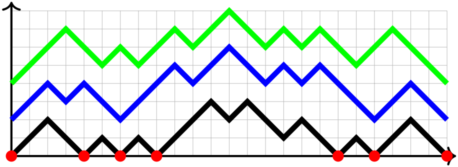

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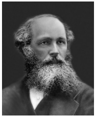
No context-free grammar, too hard model to be solved? $\Rightarrow$ Christian "I have a formula":

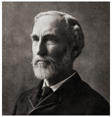
$$f_n(q) = \frac{(n-1)! \prod_{i=0}^{m-1}(2i+1)! \prod_{i=0}^{m-2}(2n+2i)!}{\prod_{i=0}^{2m-2}(n+i)!} \sum_{\ell=2}^{n+1} \binom{2n-\ell}{n-1} \binom{\ell+2m-3}{\ell-2} q^{\ell}.$$

This implies that the mean has a phase transition at $q = 2$!
In this talk, we analyse the universal phenomenon behind it,
and give the associated limit laws.

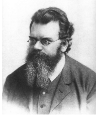
The founders of statistical mechanics:



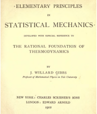
James Clerk Maxwell (1831–1879)



Josiah Willard Gibbs (1839–1903)



Ludwig Boltzmann (1844–1906)



ELEMENTARY PRINCIPLES
IN
STATISTICAL MECHANICS
DEVELOPED WITH ESPECIAL REFERENCE TO
THE RATIONAL FOUNDATION OF THERMODYNAMICS
BY
J. WILLARD GIBBS
Professor of Mathematical Physics in Yale University
NEW YORK: CHARLES SCRIBNER'S SONS
LONDON: EDWARD ARNOLD
1902

1902

Partition function $Z(1/T) = \mathrm{Tr}(\exp(-\frac{1}{T}H)) = \sum_j \exp(-\frac{E_j}{k_B T})$.

Z= Zustandssumme= state sum

Phase transition: in large structures, a continuous small variation of a parameter leads to a macroscopic change.

$\approx$ singularity of the generating function!

Definition (Gibbs distribution)

Let a family $\mathcal{C}$ of combinatorial objects and a statistic $\mathcal{X}: \mathcal{C} \to \mathbb{N}$ be given.
For real $q > 0$, the Gibbs distribution of this statistic satisfies:

$$\mathbb{P}(X_n(q) = k) = \frac{f_{n,k} q^k}{f_n(q)}, \quad k \geq 0.$$

In terms of the probability generating function $p(v) = \mathbb{E}(v^{X_n(1)})$, we have $\mathbb{E}(v^{X_n(q)}) = \frac{p(vq)}{p(q)}$.

Ex. 1: $q = 1$: uniform distribution.
Ex. 2: the Mallows distribution on permutations counting inversions [Mallows1957].
In general, we consider:

$$F(z,q) = \sum_{C \in \mathcal{C}} z^{|C|} q^{\mathcal{X}(C)} = \sum_{n \geq 0} f_n(q) z^n = \sum_{n \geq 0} \sum_{k \geq 0} f_{n,k} z^n q^k.$$

Caveat: q is not symbolic, but a weight $\in \mathbb{R}_+$. Shares the spirit of the Boltzmann distribution used in Boltzmann sampling method [Duchon, Flajolet, Louchard, Schaeffer 2004]:

$$\mathbb{P}(X_n(q) = k) = \frac{f_{n,k} q^k z^n}{F(z,q)}, \qquad \text{(thus Boltzmann} \neq \text{Gibbs)}$$

where q and z are then tuned to minimize the number of rejection in the algorithm.

$q^k = \frac{\exp(-k/T)}{Z(1/T)}$, where T is the temperature of the model.
$T \to 0 \rightsquigarrow$ frozen “solid” phase (often leading to a discrete distribution),
$T \to +\infty \rightsquigarrow$ “gaseous” phase (often leading to a Gaussian distribution),
$T = T_c \rightsquigarrow$ “liquid” phase (*where the wild things are*: unexpected fancy distribution).

(c) Maurice Sendak, 1963

Ubiquity of compositions schemes in combinatorics

Combinatorial structure = assemblage of basic building blocks

- random walks
- Pólya urns
- Galton–Watson processes
- trees
- permutations
- random mappings
- set partitions
- integer partitions
- tilings
- graphs
- maps
- ...

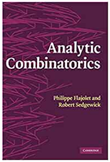
A composition scheme for generating functions

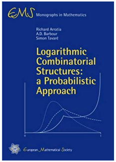
$$\sum_{n\geq 0} f_n z^n = F(z) = G\big(H(z)\big)M(z)$$

Let ρ_G and ρ_H be the radii of convergence of $G(z)$ and $H(z)$, resp. Then, the composition scheme is *critical* if $H(\rho_H) = \rho_G$ and $\rho_M \geq \rho_H$.

Examples:

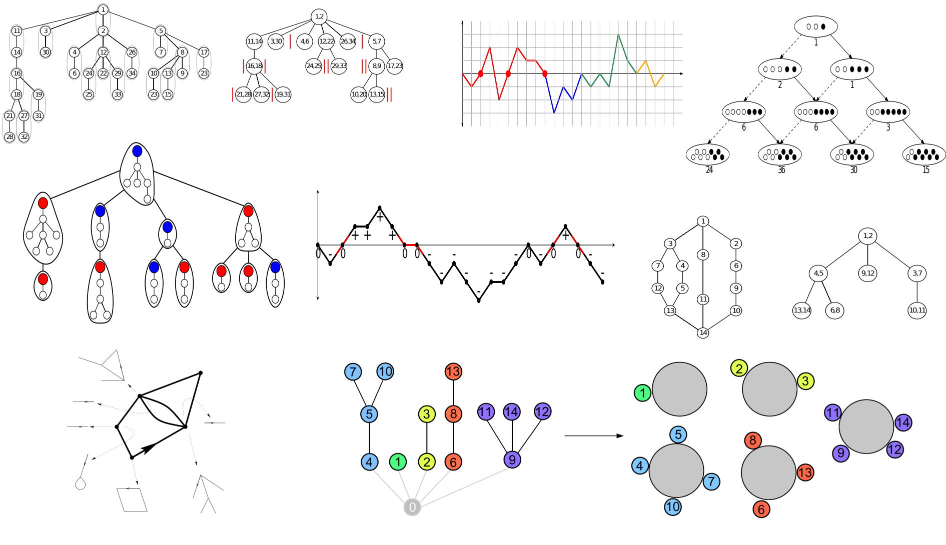
- Bicolored supertrees: $F(z) = C(2zC(z))$
- Factorization of walks: $W(z) = \frac{1}{1-H(z)} M(z)$





NB: If not critical: [Bender 1973, Gourdon 1998, Hwang 1999, ...]

Combinatorial structures $G(H(z)) \times M(z)$



here, sum of almost iid $\rightsquigarrow$ asymptotics distributions which are NON Gaussian.

Number of $\mathcal{H}$-components: Define the discrete random variable X_n of the *core size*:

$$\mathbb{P}\{X_n = k\} = \frac{[z^n u^k]F(z,u)}{[z^n]F(z,1)}$$

Note that $H(z)$ has typically the following singular expansion

$$H(z) = \tau_H + c_H\left(1 - \frac{z}{\rho_H}\right)^{\lambda_H} + \dots .$$

$\Rightarrow$ the asymptotic behaviour of $\mathbb{P}\{X_n = k\}$ depends on the *singular exponent* λ_H!

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Limit law of X_n related to certain distributions:

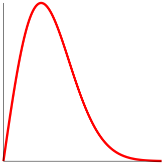
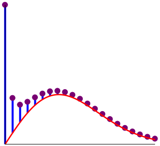
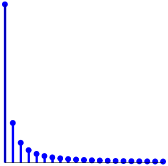
- $\lambda_H < 0$: scheme *not* critical as $H(z)$ diverges at $z = \rho_H$
  (called supercritical, typically Gaussian)
- $0 < \lambda_H < 1$: generalized Mittag-Leffler distribution
  [Banderier, Kuba, Wallner, 2021]
  ($\lambda_H = 1/2$, $M(z) = 1$: Rayleigh distribution, [Drmota, Soria 1997])
- $1 < \lambda_H < 2$: related to stable laws of parameter λ_H
  ($\lambda_H = 3/2$, $M(z) = 1$: map-Airy distribution
  [Banderier, Flajolet, Schaeffer, Soria 2001])
- $\lambda_H > 2$: Gaussian

Summary of the phase transitions for the uniform distribution (AofA2023)

Composition scheme $$F(z,u) = G(uH(z)) \cdot M(z),$$

for $F/G/H/M$ analytic at the origin, with nonnegative coefficients, and singular exponents $\lambda_F/\lambda_G/\lambda_H/\lambda_M$, such that $0 < \lambda_H < 1$.

Limit law of the number of $\mathcal{H}$-component is:

Singular exponent	$\lambda_M > \lambda_G\lambda_H$ (pure scheme)	$\lambda_M = \lambda_G\lambda_H$ (confluent scheme)	$\lambda_M < \lambda_G\lambda_H$ (degenerate scheme)
Limit law	continuous (gen. ML)	linear combination (ML $+ \mathcal{B}$)	discrete (Boltzmann $\mathcal{B}$)
Example			
	$X_n \sim Cn^{\lambda_H}\,\mathrm{ML}$	$X_n \sim \mathrm{LinComb}(n^{\lambda_H}\,\mathrm{ML}, \mathcal{B})$	$\mathbb{P}\{X_n = k\} \sim \frac{g_k \rho_G^k}{G(\rho_G)}$

See also [Stufler2022] for an approach using probability theory.

Lemma (Nature and asymptotics of q-enumerated composition schemes)

The scheme $F(z,q) = G(qH(z))$ with singular exponents $\lambda_G < 0$ and $0 < \lambda_H < 1$, has a phase transition at $q_c := \frac{\rho_G}{\tau_H} = \frac{\rho_G}{H(\rho_H)} > 0$:

- *for $0 < q < q_c$, the scheme is subcritical;*
- *for $q = q_c$, the scheme is critical;*
- *for $q > q_c$, the scheme is supercritical.*

Accordingly, if one imposes a Gibbs measure on the number of $\mathcal{H}$-components, this impacts the asymptotics of their q-enumeration $f_n(q)$ as follows:

$$f_n(q) \sim \begin{cases} \frac{c_H q G'(q\tau_H)}{\Gamma(-\lambda_H)} \rho_H^{-n} n^{-\lambda_H - 1}, & \text{for} \quad 0 < q < q_c, \\ c_G \left(-\frac{c_H}{\tau_H}\right)^{\lambda_G} \frac{1}{\Gamma(-\lambda_H \lambda_G)} \rho_H^{-n} n^{-\lambda_H \lambda_G - 1}, & \text{for} \quad q = q_c, \\ c_G \left(\frac{q\rho H'(\rho)}{\rho_G}\right)^{\lambda_G} \frac{1}{\Gamma(-\lambda_G)} \rho^{-n} n^{-\lambda_G - 1}, & \text{for} \quad q > q_c, \end{cases}$$

where, in the last case, ρ is the unique solution of $qH(\rho) = \rho_G$ in the interval $(0, \rho_H)$.

Proof.

Pringsheim's theorem on $G(qH(z))$, composition of Puiseux expansions, analyticity in some delta-domain, singularity analysis. □

Main theorem: Gibbs models and phase transitions with respect to q

Theorem

For $F(z, vq) = G\big(qvH(z)\big)$, with singular exponents $\lambda_G < 0$ and $0 < \lambda_H < 1$, the Gibbs distribution of $X_n(q)$ has (for $n \to +\infty$) a phase transition at $q_c = \frac{\rho_G}{\tau_H}$:

Parameter q	$0 < q < q_c$	$q = q_c$	$q > q_c$
Regime	*subcritical*	*critical*	*supercritical*
Singular exponent	Z^{λ_H}	$Z^{\lambda_G \lambda_H}$	Z^{λ_G}
Limit law	*discrete (Boltzmann)*	*continuous (Mittag-Leffler)*	*continuous (Gaussian)*

- *In the subcritical regime $0 < q < q_c$, the random variable $X_n - 1$ converges to a discrete distribution, a Boltzmann distribution $\mathcal{B}_{G'}(q\tau_h)$ with explicit probability generating function given by:*

$$\mathbb{P}(X_n - 1 = k) \to [v^k] \frac{G'(vq\tau_H)}{G'(q\tau_H)}.$$

In particular, if $G(z) = \frac{1}{(1-z)^m}$, the limit law of $X_n - 1$ is a negative binomial distribution $NegBin(m+1, 1-q\tau_H)$, where $X \sim NegBin(r, p)$ is defined by $\mathbb{P}(X = k) = \binom{k+r-1}{k} p^r (1-p)^k$ for $k \geq 0$.

Theorem

For $F(z, vq) = G(qvH(z))$, with singular exponents $\lambda_G < 0$ and $0 < \lambda_H < 1$, the Gibbs distribution of $X_n(q)$ has (for $n \to +\infty$) a phase transition at $q_c = \frac{\rho_G}{\tau_H}$:

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Singular exponent	Z^{λ_H}	$Z^{\lambda_G \lambda_H}$	Z^{λ_G}
Limit law	discrete (Boltzmann)	continuous (Mittag-Leffler)	continuous (Gaussian)

- *In the critical regime $q = q_c$,*

$$\frac{-c_H}{\tau_H}\frac{X_n}{n^{\lambda_H}} \xrightarrow{d} \mathrm{ML}(\alpha, \beta),$$

a Mittag-Leffler distribution (with $\alpha := \lambda_H$ and $\beta := -\lambda_G \lambda_H$) of density $f(x) = \frac{\Gamma(\beta+1)}{\alpha\Gamma(\frac{\beta}{\alpha}+1)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\Gamma(-n\alpha)} x^{n+\beta/\alpha-1}$.
In particular, for $\lambda_G = -1$ and $\lambda_H = \frac{1}{2}$, we get the Rayleigh distribution $\mathcal{R}(\sqrt{2})$.
NB: $\mathcal{R}(\sigma)$ has density $\frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}$ for $x \geq 0$.

Main theorem: Gibbs models and phase transitions with respect to q

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Singular exponent	Z^{λ_H}	$Z^{\lambda_G \lambda_H}$	Z^{λ_G}
Limit law	discrete (Boltzmann)	continuous (Mittag-Leffler)	continuous (Gaussian)

- *In the supercritical regime $q > q_c$,*
  $(X_n - \mu_n)/\sigma_n \xrightarrow{d} \mathcal{N}(0,1)$, *with linear mean and variance :*

$$\mu_n \sim \frac{\rho_G}{q\rho H'(\rho)} \cdot n, \quad \sigma_n^2 \sim \left(\frac{\rho_G^2}{q^2 \rho^2 H'(\rho)^2} - \frac{\rho_G}{q\rho H'(\rho)} + \frac{\rho_G^2 H''(\rho)}{q^2 \rho H'(\rho)^3} \right) \cdot n,$$

  where $H(\rho) = \rho_G / q$.

Main theorem: Gibbs models and phase transitions with respect to q

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Singular exponent	Z^{λ_H}	$Z^{\lambda_G \lambda_H}$	Z^{λ_G}
Limit law	*discrete (Boltzmann)*	*continuous (Mittag-Leffler)*	*continuous (Gaussian)*

- *In particular, for $n \to \infty$,*

$$\mathbb{E}(X_n) \sim \begin{cases} 1 + \frac{q\tau_H G''(q\tau_H)}{G'(q\tau_H)}, & \text{for} \quad 0 < q < q_c, \\ \frac{\lambda_G \tau_H \Gamma(-\lambda_G \lambda_H)}{c_H \Gamma((1-\lambda_G)\lambda_H)} \cdot n^{\lambda_H}, & \text{for} \quad q = q_c, \\ \frac{\rho_G}{q\rho H'(\rho)} \cdot n, & \text{for} \quad q > q_c. \end{cases}$$

Proof (sketch).

Previous lemma $\to \lim_{n\to\infty} \mathbb{E}(v^{X_n(q)}) = \lim_{n\to\infty} \frac{[z^n]F(z,qv)}{f_n(q)} = \frac{vG'(qv\tau_H)}{G'(q\tau_H)}$.

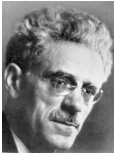
[BKW AofA2023] +moments $\Rightarrow$ ML

Hwang's quasi-power theorem $\Rightarrow$ Gaussian. □

A moment problem

Torsten Carleman

(1892-1949)



Maurice Fréchet

(1878-1973)

Theorem

For $F(z, vq) = G(qvH(z))M(z)$, for $q = q_c$, X_n converges to the 3-parameter Mittag-Leffler distribution, which is characterized by its moments

$$\mathbb{E}(\mathrm{ML}(\alpha,\beta,\gamma)^r) = \frac{\Gamma\left(r+\frac{\beta}{\alpha}\right)\Gamma(\beta+\gamma)}{\Gamma(\alpha r+\beta+\gamma)\Gamma\left(\frac{\beta}{\alpha}\right)}.$$

Proof.

Set $m_r := \mathbb{E}[X^r]$ and $m_r(n) := \mathbb{E}[X_n^r]$.

[Fréchet, Shohat 1930]: if $m_r(n) \to m_r$ then $X_n \xrightarrow{d} X$

... if the moments determine X uniquely!

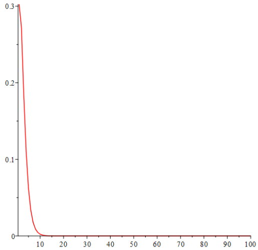
[Carleman 1923]: There is a unique distribution with such moments if :

– for support $[0, \infty)$ (Stieljes moment problem): $\sum 1/m_r^{1/2r} = \infty$

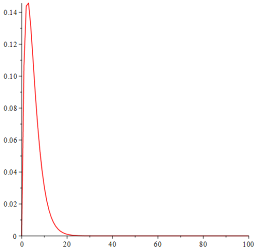
– for support $(-\infty, \infty)$ (Hamburger moment problem): $\sum 1/m_{2r}^{1/2r} = \infty$

– for support $[0, 1]$ (Hausdorff moment problem): m_r completely monotonic. □

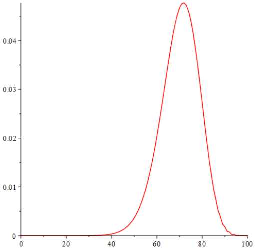
Remark: works for distributions with moments of Gamma type [Janson 2010].



$q = 1$



$q_c = 1.5$



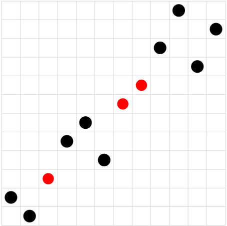
$q = 3$

The distribution (with the histogram interpolated to a curve) of returns to 0 in Motzkin excursions of length $n = 100$.

Fixed-point-biased permutations avoiding a pattern of length three

- Consider the number of fixed points in permutations of n avoiding one of the patterns 321.
- [Vella 2003, Elizalde 2004]: the generating function is

$$F(z,u) = \frac{2}{1 + 2(1-u)z + \sqrt{1-4z}}.$$



Theorem (Phase transition for fixed-point-biased permutations)

The limit Gibbs distribution of the fixed-point statistic in permutations avoiding any given pattern $p \in \{132, 321, 213\}$ has a phase transition with critical value $q_c = 3$:

Parameter q	$0 < q < 3$	$q = 3$	$q > 3$
Limit law of $X_n(q)$	$NegBin(2, 1 - q/3)$	$Rayleigh(\sqrt{2})$	$\mathcal{N}(0,1)$

$$F(z,u) = \frac{H(z)}{z} \cdot \frac{1}{1-uH(z)} = \frac{1}{uz} \cdot \frac{1}{1-uH(z)} - \frac{1}{uz}, \quad \text{where} \quad H(z) = \frac{2z}{1 + 2z + \sqrt{1-4z}}$$

See also [Chelikavada, Panzo 2023] for a more probabilistic approach.

Returns to zero in Dyck and Motzkin paths

Classical classes of paths:

- *Dyck*: steps $(1,1), (1,-1)$
- *Motzkin*: steps $(1,1), (1,-1), (1,0)$
- *Bridges* start at $(0,0)$, end at $(2n,0)$
- *Excursions* = bridges ≥ 0

Let $X_n(q)$ be the number of returns to zero in Dyck/Motzkin bridges/excursions.

Theorem (q-enumerations: limit laws for returns to zero)

Parameter q	*Limit law*
$0 < q < q_c$	$X_n - 1 \xrightarrow{d} NegBin(2, 1 - q\tau_H)$
$q = q_c$	$cst \frac{X_n}{\sqrt{n}} \xrightarrow{d} Rayleigh(\sqrt{2})$
$q > q_c$	$\frac{X_n - \mu \cdot n}{\sigma \cdot \sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$

$$q_c = \begin{cases} 2 & \text{for Dyck excursions,} \\ 1 & \text{for Dyck bridges,} \\ \frac{3}{2} & \text{for Motzkin excursions,} \\ 1 & \text{for Motzkin bridges.} \end{cases}$$

$$\text{Dyck:} \quad D(z,u) = \frac{1}{1 - z^2 u D(z)} \quad \text{and} \quad B_D(z,u) = \frac{1}{1 - 2z^2 u D(z)}$$

$$\text{Motzkin:} \quad M(z,u) = \frac{1}{1 - zu\big(1 + zM(z)\big)} \quad \text{and} \quad B_M(z,u) = \frac{1}{1 - zu\big(1 + 2zM(z)\big)}$$

Boundary contacts for quarter-plane walks

- Walks in the quarter-plane starting and ending at the origin
- Hadamard models are enumerated by a *Hadamard product* of generating functions

$$A(z) \odot B(z) := \sum_{n \geq 0} a_n b_n z^n,$$

where $A(z) = \sum_{n \geq 0} a_n z^n$ and $B(z) = \sum_{n \geq 0} b_n z^n$.

Model	Steps	Generating function $Q(z, u_1, u_2)$	Sequence Q_{2n}
Diagonal		$D(z, u_1) \odot D(z, u_2)$	$C_n \cdot C_n$
Diabolo		$D(z, u_1) \odot M(z, u_2)$	$C_n \cdot M_n$
King		$M(z, u_1) \odot M(z, u_2)$	$M_n \cdot M_n$

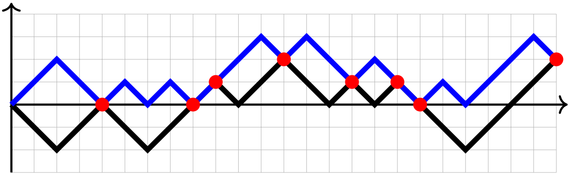
Theorem (Boundary interactions for some quarter-plane walks)

The number of axis contacts follows the NegBin/Rayleigh/Gaussian transitions phase of the previous slide.

Proof: $$\mathbb{P}(X_n(q) = k) = \frac{[z^n u^k] D(z, qu) \odot D(z)}{[z^n] D(z, q) \odot D(z)} = \frac{[z^n u^k] D(z, qu)}{[z^n] D(z, q)}$$ □

Friendly two-watermelons without wall: contacts and returns

- Friendly two-watermelons are pairs of directed walkers with steps $(1,-1)$ and $(1,1)$ that may share edges but not cross [Krattenthaler Guttmann Viennot 2000, Roitner 2020]
- A *contact* in a two-watermelon is a point (not counting the starting point) where both paths occupy the same lattice point.



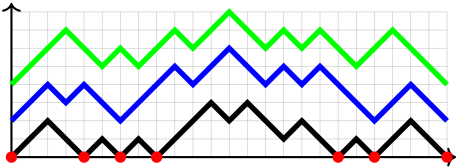
Theorem (Phase transition for contacts in friendly two-watermelons)

Parameter q	$0 < q < \frac{4}{3}$	$q = \frac{4}{3}$	$q > \frac{4}{3}$
Limit law of $X_n(q)$	$\mathit{NegBin}(2, 1-\frac{3}{4}q)$	$\mathit{Rayleigh}(\sqrt{2})$	$\mathcal{N}(0,1)$

$$F(z,u) = \frac{1}{1-u\big(z^2W(z)+2z\big)}, \qquad W(z) = \frac{1-2z-\sqrt{1-4z}}{2z^2}.$$

Number of wall contacts in watermelons

- Vicious m-watermelon of length $2n$ consists of m walkers that do not touch each other moving from $(0, 2i-2)$ to $(2n, 2i-2)$, $1 \leq i \leq m$ using steps $(1,1)$ or $(1,-1)$
- It has *a wall* if the x-axis acts as a barrier for the lowest walker
- [Krattenthaler Guttmann Viennot 2000, Krattenthaler 2006, Feierl 2009-2014]



3-watermelon with a wall of length 24 with 7 x-axis contacts

Theorem (Phase transition for wall contacts)

Parameter q	$0 < q < 2$	$q = 2$	$q > 2$
Limit law of $X_n(q)$	$NegBin(2m, 1 - \frac{q}{2})$	$\chi(2m)$	$\mathcal{N}(0,1)$

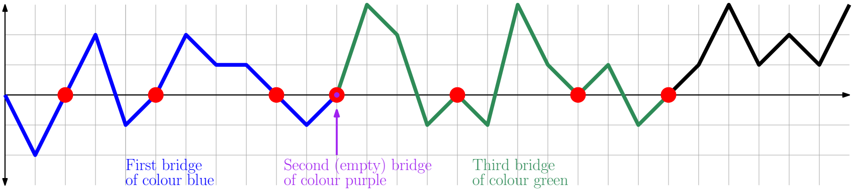
Proof: jeu de taquin + determinant $\rightsquigarrow$ Krattenthaler's huge formula, which we simplify

$$F(z,q) = \frac{q^2 z}{\sqrt{1-4z}} \cdot \frac{1}{(1 - qzC(z))^{2m}}$$

□

Open problem: bijective proof?

- An m-coloured bridge is an m-tuple $(B_1, \dots, B_m)$ of (possibly empty) bridges B_i.
- Linked to integer multicompositions [Andrews 2007, Hopkins Ouvry 2021]



A 3-coloured walk with 7 returns to zero.

Theorem (Phase transitions for returns to zero)

Parameter q	$0 < q < 1$	$q = 1$	$q > 1$
Limit law of $X_n(q)$	$\mathit{NegBin}(m, 1-q)$	$\chi(m)$	$\mathcal{N}(0,1)$

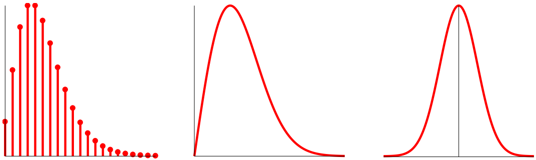
- $\chi(1)$: half-normal distribution
- $\chi(2)$: Rayleigh distribution
- $\chi(3)$: Maxwell distribution
- $\chi(m) = \frac{1}{\sqrt{2}} \mathrm{ML}(\frac{1}{2}, \frac{m}{2}, \frac{1}{2})$

$$F(z,q) = \frac{W(z)}{B(z)} \frac{1}{(1 - qA(z))^m}$$

Conclusion

- ✓ unified the analysis of the Gibbs model, under the umbrella of composition schemes
- ✓ explained the universality hidden behind some phase transitions up to now sporadically observed in the literature
- ✓ established universal limit laws (Boltzmann, Mittag-Leffler)
- ✓ Mittag-Leffler: $\mathbb{E}(\mathrm{ML}(\alpha,\beta,\gamma)^r) = \frac{\Gamma\left(r+\frac{\beta}{\alpha}\right)\Gamma(\beta+\gamma)}{\Gamma(\alpha r+\beta+\gamma)\Gamma\left(\frac{\beta}{\alpha}\right)}$
- ✓ variety of examples

Parameter q	$0 < q < q_c$	$q = q_c$	$q > q_c$
Regime	subcritical	critical	supercritical
Singular exponent	Z^{λ_H}	$Z^{\lambda_G \lambda_H}$	Z^{λ_G}
Limit law	discrete (Boltzmann)	continuous (Mittag-Leffler)	continuous (Gaussian)



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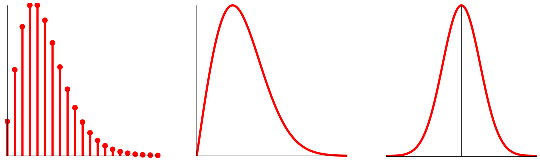

Thanks(Thanks)!