Limit theorems for the q-Pólya urn

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Conference in memory of
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Based on the work:

Limit behavior of the q-Polya urn. With D. Kouloumpou. The Ramanujan Journal 60 (1) (2023), 69-93.

Pólya's urn (1923)

Urn with black and white balls

Step 1: Pick a ball at random.

Step 2: Return the ball in the urn together with a ball of the same color.

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A_n : number of white balls after n steps

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$$\frac{A_n}{A_n + B_n} \to L \sim Beta(r, s)$$

for $n \to \infty$ with probability 1.

q-Pólya urn

Let $q \in (0, 1)$
For $x \in \mathbb{R}$

$$[x]_q := \frac{q^x - 1}{q - 1}$$

the q-analog of x. $[x]_q \to x$ for $q \to 1$.

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Urn with r white and s black balls.

- Pick at random a color with probabilities

$$\mathbf{P}_q(\text{white}) = \frac{[r]_q}{[r+s]_q} = \frac{1 - q^r}{1 - q^{r+s}}$$
$$\mathbf{P}_q(\text{black}) = q^r \frac{[s]_q}{[r+s]_q}$$

- Add a ball of this color in the urn.

Experiment realizing the probabilities:
Place the balls in order, first the white then the black.

$$\overbrace{WWW\cdots WW}^{r}\,\overbrace{BBB\cdots BBBB}^{s}$$

- Start from left, visit each ball and pick it with probability $1-q$.
- If no ball is picked out of the $r+s$, start again from left.
- When a ball is picked, the step is finished.

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$$\begin{aligned}\mathbf{P}(\text{white}) &= \overbrace{(1-q^r)}^{\text{first round}} + \overbrace{q^{r+s}(1-q^r)}^{\text{second round}} + q^{2(r+s)}(1-q^r) + \cdots \\ &= (1-q^r)\sum_{j=0}^{\infty}(q^{r+s})^j = \frac{1-q^r}{1-q^{r+s}}\end{aligned}$$

Urn giving priority to white balls

Theorem (D. C., D. Kouloumpou, 2023) Let $q \in (0,1)$

$$X_n = \#\text{extractions of black balls in first } n \text{ extractions}$$

Assume $r \geq 1$.
With probability 1, X_n is finally constant.

$X_\infty := \lim_{n\to\infty} X_n$ has probability mass function

$$f(k) = q^{rk} \begin{bmatrix} s+k-1 \\ k \end{bmatrix}_q \prod_{j=0}^{s-1} (1 - q^r q^j)$$

for each $k \in \mathbb{N}$. Negative q-binomial of the second kind with parameters s, q^r, q.

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$$\begin{bmatrix} x \\ k \end{bmatrix}_q := \frac{[x]_q [x-1]_q \cdots [x-k+1]_q}{[1]_q [2]_q \cdots [k]_q}$$

Simple argument for $\mathbf{P}(X < \infty) = 1$

$B_n :=$ number of white balls after n extractions

$$B_n \geq Y_1 + Y_2 + \cdots + Y_n$$

Y_i: i.i.d. Bernoulli$(1 - q)$

$$E_n := \{\text{black in the } n\text{-th extraction}\}$$

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$$\mathbf{P}(E_{n+1}) = \mathbf{E}\{\mathbf{P}(E_{n+1}|B_n)\} \leq \overbrace{\mathbf{E}(q^{B_n})}^{\text{no white in first round}} \leq (\mathbf{E}\{q^{Y_1}\})^n$$

$\sum_{n=1}^{\infty} \mathbf{P}(E_n) < \infty$

Apply 1st Borel-Cantelli.

The proof

The pmf of X_n is known

$$\mathbf{P}(X_n = k) = q^{rk} \frac{\begin{bmatrix} s+k-1 \\ k \end{bmatrix}_q \begin{bmatrix} r+n-k-1 \\ n-k \end{bmatrix}_q}{\begin{bmatrix} r+s+n-1 \\ n \end{bmatrix}_q}$$

$\lim_{n\to\infty} \mathbf{P}(X_n = k)$ can be calculated using

$$\lim_{n\to\infty} \begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

X_∞ =total number of black balls added

$$\mathbf{E}(X_\infty) = \sum_{j=r}^{r+s-1} \frac{q^j}{1-q^j} \sim \frac{q^r}{1-q^r}\frac{1-q^s}{1-q}$$

$T_f :=$ largest draw that gives black ball

$$\mathbf{P}(T_f \geq n) \sim q^n$$

Next: Functional limit theorems

The basic paradigm

$(X_i)_{i\geq 1}$: i.i.d with $\mathbf{E}(X_1) = 0, \mathrm{Var}(X_1) = 1$.

$S_k := X_1 + \cdots + X_k$

Strong law of large numbers: With probability 1, as $n \to \infty$,

$$\frac{S_n}{n} \to 0$$

Central limit theorem:

$$\sqrt{n}\frac{S_n}{n} = \frac{S_n}{\sqrt{n}} \Rightarrow Z \sim N(0,1)$$

Donsker's theorem. Functional CLT :

$$\left(\frac{S_{nt}}{\sqrt{n}}\right)_{t\geq 0} \Rightarrow (W_t)_{t\geq 0} \leftarrow \text{Brownian motion}$$

A functional CLT for the q-Pólya urn

$c \in (0,1), a > 0$ fixed.
For each $n \in \mathbb{N}^+$ a different q-urn

$$q_n := c^{1/n} \in (0,1) \qquad \to 1 \text{ as } m \to \infty$$

$$A^{(n)}(0) = [an] \qquad \text{initial number of white balls}$$

$$B^{(n)}(0) = w_0 \qquad \text{initial number of black balls}$$

$A^{(n)}(k) :=$ #white balls after k steps
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Theorem (D. C. , D. Kouloumpou, 2023)
As $n \to \infty$

$$\{B^{(n)}([nt]) - B^{(n)}(0)\}_{t \geq 0} \Rightarrow \{Z(t)\}_{t \geq 0}$$

Z a pure birth process inhomogeneous in time

Rates

$$\lambda_{t,j} = \frac{w_0 + j}{(1/c)^{a+t} - 1} \log(1/c)$$

[Meaning of rate

$$\mathbf{P}(Z(t+h) - Z(t) = 1 | Z(t) = j) = \lambda_{t,j} h + o(h)] \text{ as } h \to 0^+]$$

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Transition probabilities. For $0 \le s < t$,

$$Z(t) - Z(s) \Big| Z(s) = j \sim \text{NB}\left(w_0 + j, \frac{1 - c^{a+s}}{1 - c^{a+t}} \right)$$

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In particular

$$Z(t) \Rightarrow w_0 + NB(w_0, 1 - c^a) \text{ as } t \to \infty$$

Other process limits when

$\bullet A^{(n)}(0) \sim an \quad B^{(n)}(0) \to \infty$ and $B^{(n)}(0)/n \to 0.$
$\bullet A^{(n)}(0) \sim an \quad B^{(n)}(0) \sim bn, \quad a, b > 0.$

Thank you!