On the use of generating functions for studying Markov chains with binomial transitions and applications

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Markov chains with binomial transitions

- Evolution of population growth or queueing models with synchronized actions → Markov chains with binomial transitions.
- At certain time points (service completions, abandonment epochs, etc) the present customers/units decide **independently** whether to leave the system or not, with **the same probability** p.
  →
  The number of customers is reduced according to a binomial distribution.
- Transition rates:

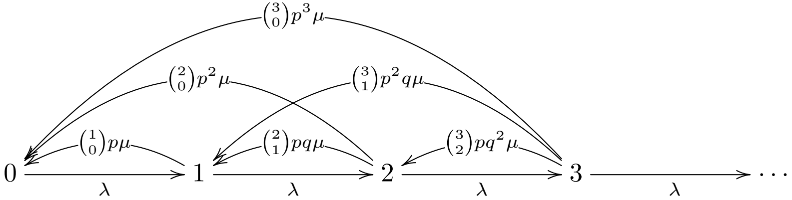
$$n \to m \quad : \quad \text{rate} \times \binom{n}{m} p^{n-m}(1-p)^m = \binom{n}{m} p^{n-m} q^m .$$

- System with synchronization ↦ state inhomogeneous Markov chains.
- Synchronization ↦ Binomial rates **dependent on the state** n.

Outline

The Poisson process with **binomial catastrophes**

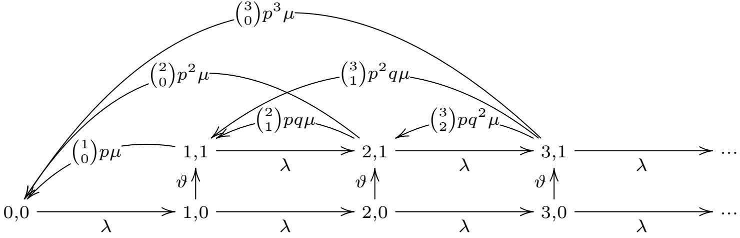
- Poisson immigration process at rate λ.
- Catastrophes occur at rate μ according to a renewal process.
- At a catastrophe epoch each unit is destroyed with probability p, or survives with probability $q = 1 - p$, independently of the others.
- $N(t)$: number of units in the system at time t (in case where the catastrophe process is Poisson).
- $\{(N(t) : t \geq 0)$ is a CTMC with diagram:



- Economou JAP
- Artalejo, Economou and Lopez-Herrero MBE

A model with setup times and **synchronized services**

- Poisson arrival process at rate λ.
- Single server, who serves simultaneously all present customers. The successive service times follow $\mathrm{Exp}(\mu)$.
- At a service completion epoch, each customer is satisfied and departs with probability p or repeats his service with $q = 1 - p$
- Empty system → deactivation of the server.
- Arrival at an empty system → Setup time∼$\mathrm{Exp}(\vartheta)$.
- $N(t)$: number of customers in the system at time t.
- $I(t)$: state of the server at time t (0=off and 1=on).
- $\{(N(t), I(t)) : t \geq 0\}$ is a CTMC with diagram:



- Economou and Kapodistria, PEIS.

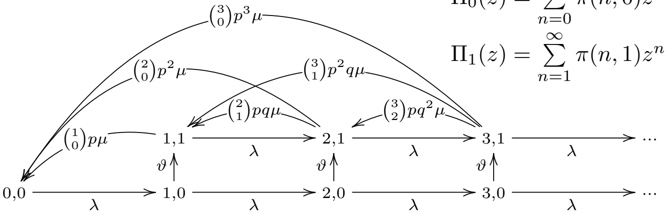
Analysis

- Study of the steady-state behavior (stationary probabilities/moments: exact formulas or recursive schemes).
- Study of the times from a given state to an empty system (busy periods, population cycles).
- Study of the limiting behavior in the case of high rate of synchronization events.
- Bounds, approximations etc.

The model with setup times and synchronized services: Generating functions for the steady-state distribution

$$\Pi_0(z) = \sum_{n=0}^{\infty} \pi(n,0) z^n$$

$$\Pi_1(z) = \sum_{n=1}^{\infty} \pi(n,1) z^n$$



- Balance equations:

$$\begin{aligned}
\lambda\pi(0,0) &= \mu \sum_{n=1}^{\infty} p^n \pi(n,1) = \mu\Pi_1(p) \\
(\lambda+\vartheta)\pi(n,0) &= \lambda\pi(n-1,0), \quad n \geq 1 \\
(\lambda+\mu)\pi(1,1) &= \vartheta\pi(1,0) + \mu \sum_{j=1}^{\infty} \binom{j}{1} p^{j-1} q \pi(j,1) \\
(\lambda+\mu)\pi(n,1) &= \vartheta\pi(n,0) + \lambda\pi(n-1,1) + \mu \sum_{j=n}^{\infty} \binom{j}{n} p^{j-n} q^n \pi(j,1)\,, \quad n \geq 2.
\end{aligned}$$

The model with setup times and synchronized services: Generating functions for the steady-state distribution

- Transforming the balance equations for states $(n,0)$ to a single equation for $\Pi_0(z)$:

$$
\begin{aligned}
{\color{blue}z^0\times}\quad \lambda\pi(0,0) &= \mu\sum_{n=1}^{\infty} p^n\pi(n,1) = \mu\Pi_1(p) \\
{\color{blue}z^n\times}\quad (\lambda+\vartheta)\pi(n,0) &= \lambda\pi(n-1,0), \quad n\geq 1 \\
&\Downarrow \sum \\
(\lambda+\vartheta)\Pi_0(z)-(\lambda+\vartheta)\pi(0,0) &= \lambda z\Pi_0(z) \\
&\Downarrow \\
\Pi_0(z) &= \frac{\lambda+\vartheta}{\lambda+\vartheta-\lambda z}\pi(0,0)
\end{aligned}
$$

The model with setup times and synchronized services: Generating functions for the steady-state distribution

- Transforming the balance equations for states $(n,1)$ to a single equation for $\Pi_0(1)$:

$$
\begin{aligned}
z\times \quad (\lambda+\mu)\pi(1,1) &= \vartheta\pi(1,0)+\mu\sum_{j=1}^{\infty}\binom{j}{1}p^{j-1}q\pi(j,1)\\
z^n\times \quad (\lambda+\mu)\pi(n,1) &= \vartheta\pi(n,0)+\lambda\pi(n-1,1)\\
&\quad +\mu\sum_{j=n}^{\infty}\binom{j}{n}p^{j-n}q^n\pi(j,1)\,,\quad n\geq 2.\\
&\Downarrow \quad \sum\\
(\lambda+\mu)\Pi_1(z) &= \vartheta(\Pi_0(z)-\pi(0,0)+\lambda z\Pi_1(z)+\mu\Pi_1(1-q+qz)\\
&\Updownarrow\\
(\lambda+\mu-\lambda z)\Pi_1(z) &= \mu\Pi_1(1-q+qz)+\frac{\lambda(\lambda+\vartheta)}{\lambda+\vartheta-\lambda z}(z-1)\pi(0,0).\\
A(z)\Pi_1(z) &= B(z)\Pi_1(1-q+qz)+\Gamma(z).
\end{aligned}
$$

The general scheme: Generating functions for the steady-state distribution

- Transforming the balance equations of Markov chains with binomial transitions always lead to equation of type

$$A(z)\Pi(z) = B(z)\Pi(1-q+qz) + \Gamma(z).$$

(Similar to branching MC with immigration)

- The target: Solve such equations for $\Pi(z)$.
- Solving for $\Pi(z)$:

$$\Pi(z) = \frac{B(z)}{A(z)}\Pi(1-q+qz) + \frac{\Gamma(z)}{A(z)}.$$

- Setting $z := 1-q+qz$:

$$\Pi(1-q+qz) = \frac{B(1-q+qz)}{A(1-q+qz)}\Pi(1-q^2+q^2z) + \frac{\Gamma(1-q+qz)}{A(1-q+qz)}.$$

The general scheme: Generating functions for the steady-state distribution

- Plugging the expression for $\Pi(1-q+qz)$ in the formula for $\Pi(z)$ yields

$$\Pi(z) = \frac{B(z)B(1-q+qz)}{A(z)A(1-q+qz)}\Pi(1-q^2+q^2z) + \frac{B(z)\Gamma(1-q+qz)}{A(z)A(1-q+qz)} + \frac{\Gamma(z)}{A(z)}.$$

- Iterating yields:

$$\Pi(z) = \frac{\prod_{i=0}^{n} B(1-q^i+q^iz)}{\prod_{i=0}^{n} A(1-q^i+q^iz)}\Pi(1-q^{n+1}+q^{n+1}z) + \sum_{k=0}^{n}\Gamma(1-q^k+q^kz)\frac{\prod_{i=0}^{k-1} B(1-q^i+q^iz)}{\prod_{i=0}^{k} A(1-q^i+q^iz)}.$$

The general scheme: Generating functions for the steady-state distribution

- Taking $n \to \infty$ yields

$$\Pi(z) = \frac{\prod_{i=0}^{\infty} B(1-q^i+q^iz)}{\prod_{i=0}^{\infty} A(1-q^i+q^iz)}\Pi(1) + \sum_{k=0}^{\infty} \Gamma(1-q^k+q^kz)\frac{\prod_{i=0}^{k-1} B(1-q^i+q^iz)}{\prod_{i=0}^{k} A(1-q^i+q^iz)}.$$

The general scheme: Generating functions for the steady-state distribution

- What makes the scheme works?
  The fact that the n-th compositions of the transformation $S(z) = 1 - q + qz$ are easily calculable:

$$S^{\circ n}(z) = (S \circ S \circ \cdots \circ S)(z) = 1 - q^n + q^n z.$$

  The fact that $\lim_{n\to\infty} S^{\circ n}(z) = 1$, for $q \in [0, 1)$.

Applying the general scheme on the model with setup times and synchronized services

$\Pi_1(z) = \frac{B(z)}{A(z)}\Pi_1(1-q+qz) + \frac{\Gamma(z)}{A(z)}$ $\Pi_0(z) = \frac{\lambda+\vartheta}{\lambda+\vartheta-\lambda z}\pi(0,0)$	$A(z) = \lambda+\mu-\lambda z$ $B(z) = \mu$ $\Gamma(z) = \frac{\lambda(\lambda+\vartheta)}{\lambda+\vartheta-\lambda z}(z-1)\pi(0,0)$

$$\begin{aligned}\Pi_1(z) \quad &= \quad \prod_{i=0}^{\infty} \frac{\mu}{\mu+\lambda(1-z)q^i}\Pi_1(1) \\ &\quad + \frac{\lambda}{\mu}(\lambda+\vartheta)\pi(0,0)(z-1)\sum_{k=0}^{\infty}\frac{q^i}{\vartheta+\lambda(1-z)q^k}\prod_{i=0}^{k}\frac{\mu}{\mu+\lambda(1-z)q^i}\end{aligned}$$

Moreover, $\Pi_0(1)+\Pi_1(1) = 1$ which yields the value of $\pi(0,0)$ and therefore $\Pi_1(1)$.

q-hypergeometric series

- q-shifted factorial:

$$\begin{aligned}(a;q)_0 &= 1,\\ (a;q)_n &= (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),\ n\geq 1,\\ (a;q)_\infty &= \prod_{k=0}^{\infty}(1-aq^k).\end{aligned}$$

- q-analogues of the exponential function e^z:

$$\begin{aligned}e_q(z) &= \frac{1}{(z;q)_\infty},\\ E_q(z) &= (-z;q)_\infty.\end{aligned}$$

- q-hypergeometric series:

$$\begin{aligned}{}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right) &\equiv {}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;q,z)\\ &= \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}z^n,\end{aligned}$$

The model with setup times and synchronized services: Generating functions for the steady-state distribution

Theorem

The equilibrium state probability of an empty system $\pi(0,0)$ *is given by*

$$\pi(0,0) = \left[\frac{\lambda+\vartheta}{\vartheta} + \frac{\lambda}{\lambda+\mu} E_q\left(\frac{\lambda}{\mu}\right) {}_3\phi_2\left(-\frac{\lambda}{\vartheta}, q, 0; -\frac{\lambda}{\vartheta}q, -\frac{\lambda}{\mu}q; q, q\right)\right]^{-1}.$$

The partial probability generating functions $\Pi_0(z)$ *and* $\Pi_1(z)$ *are given by*

$$\begin{aligned}
\Pi_0(z) &= \frac{\lambda+\vartheta}{\lambda+\vartheta-\lambda z}\pi(0,0) \\
\Pi_1(z) &= \left[1 - \frac{\lambda+\vartheta}{\vartheta}\pi(0,0)\right] e_q\left(-\frac{\lambda}{\mu}(1-z)\right) \\
&\quad - \frac{\lambda(\lambda+\vartheta)(1-z)}{(\lambda+\vartheta-\lambda z)(\lambda+\mu-\lambda z)} \times \\
&\quad \times \pi(0,0)\, {}_3\phi_2\left(\begin{matrix} -\frac{\lambda}{\vartheta}(1-z), q, 0 \\ -\frac{\lambda}{\vartheta}q(1-z), -\frac{\lambda}{\mu}q(1-z) \end{matrix}; q, q\right).
\end{aligned}$$

The model with setup times and synchronized services: Exact formulas for steady-state moments

Theorem

The factorial moments $m_{(n)} = E[N(N-1)(N-2)\cdots(N-n+1)]$ *of the equilibrium number of customers in the system are given by*

$$\begin{aligned} m_{(n)} &= \frac{(\lambda+\vartheta)\lambda^n n!}{\vartheta^{n+1}}\pi(0,0) + \sum_{k=1}^{n} \frac{(\lambda+\vartheta)\lambda^n n!}{\vartheta^k \mu^{n-k+1}} \frac{(q;q)_{k-1}}{(q;q)_n}\pi(0,0) \\ &\quad + \frac{\lambda^n n!}{\mu^n}(q;q)_n \left[1 - \frac{\lambda+\vartheta}{\vartheta}\pi(0,0)\right], \; n \geq 1. \qquad (1) \end{aligned}$$

Proof: Applying q-hypergeometric series identities and expand the factorial moment generating function $P(z) = \Pi_0(z) + \Pi_1(z)$.

The general scheme: Recursive scheme for for steady-state moments

- We start with the 'ubiquitous' type of generating function equation for Markov chains with binomial transitions:

$$A(z)\Pi(z) = B(z)\Pi(1-q+qz) + \Gamma(z)$$

- We differentiate it n times and evaluate at $z=1$:

$$\sum_{k=0}^{n} \binom{n}{k} A^{(k)}(1)\Pi^{(n-k)}(1) = \sum_{k=0}^{n} \binom{n}{k} B^{(k)}(1) q^{n-k} \Pi^{(n-k)}(1) + \Gamma^{(n)}(1).$$

- We solve for $\Pi^{(n)}(1)$ in terms of $\Pi^{(k)}(1)$, $k = 1, 2, \ldots, n-1$.

The model with setup times and synchronized services: Time to an empty system (remaining busy period)

- $\varphi_{(n,i)}(s) = E[e^{-sT_{(n,i)}}]$: the Laplace-Stieltjes transform of a first passage time to state $(0,0)$ starting from (n,i).
- First step analysis:

$$\begin{aligned}
\varphi_{(0,0)}(s) &= 1 \\
\varphi_{(n,0)}(s) &= \frac{\lambda}{\lambda+\vartheta+s}\varphi_{(n+1,0)}(s) + \frac{\vartheta}{\lambda+\vartheta+s}\varphi_{(n,1)}(s),\ n \geq 1 \\
\varphi_{(n,1)}(s) &= \frac{\lambda}{\lambda+\mu+s}\varphi_{(n+1,1)}(s) + \frac{\mu p^n}{\lambda+\mu+s} \\
&\quad + \frac{\mu}{\lambda+\mu+s}\sum_{j=1}^{n}\binom{n}{n-j}p^{n-j}q^j\varphi_{(j,1)}(s),\ n \geq 1.
\end{aligned}$$

The model with setup times and synchronized services: Time to an empty system (remaining busy period)

- We define the mixed transforms $\Phi_0(s,z)$, $\Phi_1(s,z)$ (i.e. the generating functions of the Laplace-Stieltjes transforms):

$$\Phi_0(z) \equiv \Phi_0(s,z) = \sum_{n=0}^{\infty} \varphi_{(n,0)}(s) z^n,$$

$$\Phi_1(z) \equiv \Phi_1(s,z) = \sum_{n=1}^{\infty} \varphi_{(n,1)}(s) z^n.$$

The model with setup times and synchronized services: Time to an empty system (remaining busy period)

- We apply the standard generating function approach:
  Multiplying the equation for (n,i) by z^n and adding for all n:

$$
\begin{aligned}
z^n \times \quad \varphi_{(n,1)}(s) &= \frac{\lambda}{\lambda+\mu+s}\varphi_{(n+1,1)}(s) + \frac{\mu p^n}{\lambda+\mu+s} \\
&\quad + \frac{\mu}{\lambda+\mu+s}\sum_{j=1}^{n}\binom{n}{n-j}p^{n-j}q^j\varphi_{(j,1)}(s), \\
&\Downarrow \quad \sum_n \\
\left[(\lambda+\mu+s)z-\lambda\right]\Phi_1(s,z) &= \frac{\mu p z^2}{1-pz} - \lambda z\varphi_{(1,1)}(s) \\
&\quad + \frac{\mu z}{1-pz}\Phi_1\left(s, \frac{qz}{1-(1-q)z}\right) \\
A(z)\Phi_1(z) &= B(z)\Phi_1\left(\frac{qz}{1-(1-q)z}\right) + \Gamma(z).
\end{aligned}
$$

The general scheme: Generating functions for the Laplace-Stieltjes transforms

- Transforming the first-step analysis equations for the absorption times to an empty system for Markov chains with binomial transitions leads to equations of the form

$$A(z)\Phi_1(z) \quad = \quad B(z)\Phi_1(\frac{qz}{1-(1-q)z}) + \Gamma(z).$$

- The target: Solve such equations for $\Phi_1(z)$.

The general scheme: Generating functions for the Laplace-Stieltjes transforms

- Let

$$\begin{aligned} T_0(z) &= z \\ T_1(z) &= \frac{qz}{1-(1-q)z} \\ T_{k+1}(z) &= T_1(T_k(z)),\ k \geq 1 \end{aligned}$$

- Iteration and taking the limit works similarly to the case of the steady-state distribution.
- What makes the scheme works?
  The fact that the n-th compositions of the transformation $T(z) = \frac{qz}{1-(1-q)z}$ are easily calculable:

$$T^{\circ n}(z) = (T \circ T \circ \cdots \circ T)(z) = \frac{q^n z}{1-(1-q^n)z}.$$

  The fact that $\lim_{n\to\infty} T^{\circ n}(z) = 0$.

The general scheme: Generating functions for the Laplace-Stieltjes transforms

- Iterating the basic equation

$$A(z)\Phi_1(z) \quad = \quad B(z)\Phi_1(\frac{qz}{1-(1-q)z}) + \Gamma(z)$$

n times and taking $n \to \infty$ yields a closed formula for $\Phi_1(z)$.

Conclusions

- Unified framework for Markov chains with binomial transitions.
- Computational schemes for
  * the steady-state distribution,
  * the steady-state moments,
  * the Laplace-Stieltjes tranforms of first-passage times to empty system,

  using the theory of basic q-hypergeometric series.
- Formal convergence results for high and low level of synchronization.
- Approximations and bounds for the main performance descriptors.

References

- ADAN, I., ECONOMOU, A. AND KAPODISTRIA, S. (2008) Synchronized reneging in queueing systems with vacations.
- ALTMAN, E.,YECHIALI, U. (2006) Analysis of customers' impatience in queues with server vacations. *Queueing Syst.* **52**, no. 4, 261–279.
- ARTALEJO, J.R., ECONOMOU, A. AND LOPEZ-HERRERO, M.J. (2007) Evaluating growth measures in populations subject to binomial and geometric catastrophes. *Mathematical Biosciences and Engineering* **4**, 573–594.
- BROCKWELL, P.J., GANI, J. AND RESNICK, S.I. (1982) Birth, immigration and catastrophe processes, *Advances in Applied Probability* **14,** 709–731.
- ECONOMOU, A. (2004) The compound Poisson immigration process subject to binomial catastrophes, *Journal of Applied Probability* **41,** no. 2, 508–523.
- ECONOMOU, A. AND KAPODISTRIA, S. (2007) q-series in markov chains with binomial transitions: studying a queue with synchronization, *Probability in the Engineering and Informational Sciences.*
- NEUTS, M.F. (1994) An interesting random walk on the nonnegative integers, *Journal of Applied Probability* **31,** 48–58.

References

- GASPER, G. AND RAHMAN, M. (2004) *Basic Hypergeometric Series*, 2nd edn. Cambridge University Press.
- ISMAIL, M.E.H. (1985) A queueing model and a set of orthogonal polynomials, *Journal of Mathematical Analysis and Applications* **108,** 575–594.
- KEMP, A.W. AND NEWTON, J. (1990) Certain state-dependent processes for dichotomised parasite populations, *Journal of Applied Probability* **27,** 251–258.
- KEMP, A.W. (1992) Steady-state Markov chain models for the Heine and Euler distributions, *Journal of Applied Probability* **29,** 869–876.
- KEMP, A.W. (1998) Absorption sampling and absorption distribution, *Journal of Applied Probability* **35,** 489–494.
- KEMP, A.W. (2005) Steady-state Markov chain models for certain q-confluent hypergeometric distributions, *Journal of Statistical Planning and Inference* **135,** 107–120.