Strong laws of large numbers for lightly trimmed sums of generalized Oppenheim expansions

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Outline

Expansions of real numbers

- The **Lüroth** series (1883): Every real number $x \in (0,1]$

$$x = \frac{1}{d_1} + \frac{1}{(s_1)\, d_2} + \cdots + \frac{1}{(s_1 \cdots s_n)\, d_{n+1}} + \cdots = \sum_{k=1}^{\infty} \frac{1}{\left(\prod_{h=1}^{k-1} s_h\right) d_k}$$

where $(d_n)_{n\geqslant 1} = (d_n(x))_{n\geqslant 1}$ is a sequence of integers ≥ 2 and $s_n = d_n\,(d_n - 1)\,,\, n \geqslant 1$.

- The **Engel** series (1913): Every real number $x \in (0,1)$:

$$x = \frac{1}{d_1} + \frac{1}{d_1 d_2} + \cdots + \frac{1}{d_1 d_2 \cdots d_n} + \cdots = \sum_{k=1}^{\infty} \prod_{h=1}^{k} \frac{1}{d_k}$$

where $(d_n)_{n\geqslant 1} = (d_n(x))_{n\geqslant 1}$ is a non-decreasing sequence of positive integers uniquely defined in terms of x.

Expansions of real numbers

- The **Sylvester** series (see for example Perron (1960)): Every real number $x \in (0,1)$:

$$x = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n} + \cdots = \sum_{k=0}^{\infty} \frac{1}{d_k}$$

where $(d_n)_{n\geqslant 1} = (d_n(x))_{n\geqslant 1}$ is a sequence of positive integers uniquely defined in terms of x.

Expansions of real numbers

Oppenheim series (Oppenheim (1972)): Let $(\gamma_n)_{n\geqslant 1}$ be a sequence of positive rational-valued functions defined on $\mathbb{N}\backslash\{1\}$ and satisfying

$$\gamma_n(h) \geqslant \frac{1}{h(h-1)} \quad \text{for all } n \geqslant 1.$$

For $x \in (0,1)$, the Oppenheim expansion of x is

$$\begin{aligned} x &= \frac{1}{d_1} + \gamma_1(d_1)\frac{1}{d_2} + \cdots + \gamma_1(d_1)\cdots\gamma_n(d_n)\frac{1}{d_{n+1}} + \cdots \\ &= \sum_{k=1}^{\infty}\left\{\prod_{h=1}^{k-1}\gamma_h(d_h)\right\}\frac{1}{d_k} \end{aligned}$$

where the digits $d_n = d_n(x)$ are integers uniquely determined in terms of x.

Expansions of real numbers

Observe that for

- $\gamma_n(h) = \frac{1}{h(h-1)}$
- $\gamma_n(h) = \frac{1}{h}$
- $\gamma_n(h) = 1$

the Oppenheim expansion is reduced to the Lüroth, the Engel and the Sylvester series respectively.

Any Oppenheim expansion satisfies the below property (Galambos (1976)):

Let $(D_n)_{n\geq 1}$ be the sequence of Oppenheim digits, and define $B_n = D_n - 1$; then

$$\begin{aligned} P\left(B_{n+1}=h_{n+1} \mid B_n=h_n,\ldots,B_1=h_1\right) &= \frac{\gamma_n\left(h_n+1\right)h_n\left(h_n+1\right)}{h_{n+1}\left(h_{n+1}+1\right)} \\ &= \int_{\alpha_n}^{\beta_n} 1 \,\mathrm{d}u \\ &= \beta_n-\alpha_n = F(\beta_n)-F(\alpha_n), \end{aligned}$$

where F is the distribution function of the uniform law on $[0,1]$, $h_1,\ldots,h_n,h_{n+1}$ are positive integers and $\alpha_n := \delta_n\left(h_n, h_{n+1}+1, q_n\right)$, $\beta_n := \delta_n\left(h_n, h_{n+1}, q_n\right)$ for suitable sequences of functions $\delta_n\left(h, h', q\right)$ and $q_n := q_n\left(h_1,\ldots,h_n\right)$.

Remark: $(B_n)_{n\geq 1}$ is not necessarily a Markov chain since the q_n may depend on (some of) the integers $h_1,\ldots,h_{n-1}$.

Giuliano (2018):

Let $(D_n)_{n\geq 1}$ be the sequence of Oppenheim digits, and define $B_n = D_n - 1$; then

$$\begin{aligned} P\left(B_{n+1} = h_{n+1} \mid B_n = h_n, \ldots, B_1 = h_1\right) &= \frac{\gamma_n\left(h_n+1\right) h_n\left(h_n+1\right)}{h_{n+1}\left(h_{n+1}+1\right)} \\ &= \int_{\alpha_n}^{\beta_n} f \,\mathrm{d}u, \end{aligned}$$

where f is a density on $(0,1)$, $h_1, \ldots, h_n, h_{n+1}$ are positive integers and $\alpha_n := \delta_n\left(h_n, h_{n+1}+1, q_n\right)$, $\beta_n := \delta_n\left(h_n, h_{n+1}, q_n\right)$ for suitable sequences of functions $\delta_n\left(h, h', q\right)$ and $q_n := q_n\left(h_1, \ldots, h_n\right)$.

Let $(B_n)_{n\geq 1}$ be a sequence of integer valued random variables defined on $(\Omega, \mathcal{A}, P)$, where $\Omega = [0,1]$, $\mathcal{A}$ is the σ-algebra of the Borel subsets of $[0,1]$ and P is the Lebesgue measure on $[0,1]$.

Let $\{F_n, n \geq 1\}$ be a sequence of probability distribution functions with $F_n(0) = 0$, for all n and moreover let $\varphi_n : \mathbb{N}^* \to \mathbb{R}^+$ be a sequence of functions.

Furthermore, let $(q_n)_{n\geq 1}$ with $q_n = q_n(h_1, \ldots, h_n)$ be a sequence of nonnegative numbers (i.e. possibly depending on the n integers $h_1, \ldots, h_n$) such that, for $h_1 \geq 1$ and $h_j \geq \varphi_{j-1}(h_{j-1})$, $j = 2, \ldots, n$ we have

$$P\big(B_{n+1} = h_{n+1} \mid B_n = h_n, \ldots, B_1 = h_1\big) = F_n(\beta_n) - F_n(\alpha_n),$$

where

$$\alpha_n = \delta_n(h_n, h_{n+1}{+}1, q_n), \quad \beta_n = \delta_n(h_n, h_{n+1}, q_n) \quad \text{with} \quad \delta_j(h,k,q) = \frac{\varphi_j(h)(1+q)}{k + \varphi_j(h)q}.$$

Let $Q_n = q_n(B_1, \ldots, B_n)$ and define

$$R_n = \frac{B_{n+1} + \varphi_n(B_n) Q_n}{\varphi_n(B_n)(1 + Q_n)} \quad \text{and} \quad S_n = \sum_{i=1}^{n} R_i.$$

For $f = 1$:

- For $Q_n = 0$: Classical Oppenheim scheme i.e. $R_n = \frac{B_{n+1}}{\varphi_n(B_n)}$. Different choices of φ_n lead to ratios of functions for the Lüroth, Engel and Sylvester random digits.
- For $Q_n \geq 0$: Classical and Oppenheim continued fraction expansions.

Note: Depending on the choice of φ_n and q_n the dependence structure may vary.

Theorem (Giuliano and Hadjikyriakou (2020))

For any integer n and for $x \geq 1$,

$$E\left[F_n\left(\frac{\varphi_n(B_n)(1+Q_n)}{x\varphi_n(B_n)(1+Q_n)+1}\right)\right] \leq P(R_n > x) \leq F_n\left(\frac{1}{x}\right).$$

Moreover, if $\varphi_n \geq 1$

$$F_n\left(\frac{1}{x+1}\right) \leq P(R_n > x) \leq F_n\left(\frac{1}{x}\right).$$

i.e. for $U_n \sim F_n$ for every n, R_n is stochastically dominated by U_n^{-1}.

Notice that for $F_n \equiv x$ (the uniform law), R_n do not have finite moments thus, existence of means is not assumed in any of the results.

Theorem (Giuliano and Hadjikyriakou (2024))

Let $\varphi_n \geq 1$ for every integer n. Then, for every k, every finite sequence of integers $i_1, \ldots, i_k$ and every finite sequence of numbers $x_1, \ldots, x_k \geq 1$ we have

$$\prod_{j=1}^{k} F_{i_j}\left(\frac{1}{x_j+1}\right) \leq P(R_{i_1} > x_1, \ldots, R_{i_k} > x_k) \leq \prod_{j=1}^{k} F_{i_j}\left(\frac{1}{x_j}\right).$$

Proposition (Giuliano and Hadjikyriakou (2024))

The random variables $(R_n)_{n\geq 1}$ have a *long-tailed* distribution.

Theorem (Giuliano and Hadjikyriakou (2020))

Let $(R_n)_{n\geq 1}$ be independent random variables and let the distribution functions $(F_n)_{n\geq 1}$ to satisfy

$$\lim_{t\to 0}\sup_n \left|\frac{F_n(t)}{t} - c\right| = 0.$$

Then for every $b > 2$,

$$\lim_{n\to\infty} \frac{1}{\log^b n}\sum_{k=1}^{n} \frac{\log^{b-2} k}{k} R_k = \frac{1}{b} \quad a.s.$$

Proposition (Giuliano and Hadjikyriakou (2020))

Let $(R_n)_{n\geq 1}$ be as defined above such that $\forall h_1, \ldots, h_n$, $\varphi_n(h_n) = c_n$ and $q_n = q_n(h_1, \ldots, h_n) = d_n$. Then, the sequence $(R_n)_{n\geq 1}$ consists of independent random variables.

Remarks:

- No assumptions on F_n were necessary.
- For $F_n = U[0,1]$, $\varphi_n(h_n) \equiv 1$ and $q_n \equiv 0$, R_n reduces to the Lüroth series expansion.

Theorem (Giuliano and Hadjikyriakou (2020/23))

Let $(R_n)_{n\geq 1}$ be as defined above with $\varphi_n \geq 1$ for every n and consider F_n for which $\exists M < \infty$ such that $\forall j = 1, \ldots, n$

$$F_j(x) - F_j(y) \leq M(x-y) \text{ for } x > y$$

and there exists $c > 0$ for which

$$\lim_{t\to 0} \sup_n \left| \frac{F_n(t)}{t} - c \right| = 0.$$

Then,

1. for $\gamma > 1$, $\frac{1}{n^\gamma} \sum_{k=1}^{n} \frac{R_k}{k} \to 0,$ a.s. (2020)
2. for $\beta > 0$, $p \geq 2$ and $\rho(n)$ such that $\sum_{n=1}^{\infty} 1/\rho(n)^2 < \infty$

$$\frac{1}{\rho(n)\log^\beta n} \sum_{j=1}^{n} \frac{\log^{\beta-p} j}{j} R_j \to 0 \quad \text{a.s.} \quad (2023)$$

Theorem (Giuliano and Hadjikyriakou (2023))

Assume that there exists $M < \infty$, $\alpha > 0$ and $L > 0$ such that

(i)

$$F_j(x) - F_j(y) \leq M(x - y), \quad \text{for} \quad x > y, \quad \forall j = 1, 2, \ldots, n$$

(ii)

$$\lim_{x \to \infty} \sup_n \left| \frac{F_n(x)}{x^\alpha} - L \right| = 0$$

and that the sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ satisfy

$$\sum_{j=1}^{n} a_j^\alpha = o\left(b_n^\alpha\right) \quad \text{and} \quad n/b_n^{p-1} \to 0 \quad n \to \infty \quad \text{for some} \quad p > 1.$$

Then, for $R_{nj} = R_j I\left(R_j \leq \frac{b_n}{a_j}\right) + \frac{b_n}{a_j} I\left(R_j > \frac{b_n}{a_j}\right)$.

$$\frac{1}{b_n^p} \sum_{j=1}^{n} a_j \left(R_j - ER_{nj}\right) \xrightarrow{P} 0 \quad n \to \infty$$

Theorem (Giuliano and Hadjikyriakou (2023))

Assume that the conditions of the previous Theorem are satisfied. Then,

(a) if $\{D_n\}_{n\geq 1}$ is assumed to be the Lüroth sequence, and $\alpha = 1$,

$$\frac{1}{b_n^p}\sum_{j=1}^{n} a_j D_{j+1} \xrightarrow{P} 0, \quad n \to \infty$$

(b) if $\{D_n\}_{n\geq 1}$ is assumed to be the Engel's sequence,

$$\frac{1}{b_n^p}\sum_{j=1}^{n} a_j \frac{D_{j+1}}{D_j} \xrightarrow{P} 0, \quad n \to \infty$$

(c) if $\{D_n\}_{n\geq 1}$ is assumed to be the Sylvester's sequence,

$$\frac{1}{b_n^p}\sum_{j=1}^{n} a_j \frac{D_{j+1}}{D_j^2} \xrightarrow{P} 0, \quad n \to \infty$$

- Since $(R_n)_{n\geq 1}$ do not have finite expectations a strong law for the quantity $\frac{1}{a_n}\sum_{i=1}^{n} R_i$ cannot be proven.
- However, weak laws are feasible.

  Giuliano (2018), under some conditions for the involved distributions, proved the convergence in probability of $\frac{1}{n\log n}\sum_{i=1}^{n} R_i$.
- **Question**: Can we prove a strong law of large numbers, after deleting finitely many of the largest summands from the partial sums?

We define the sequence of "trimmed" partial sums as

$$^{(r)}S_n = \sum_{i=1}^{n} R_i - \sum_{k=1}^{r} M_n^{(k)}$$

for $M_n^{(k)}$ denoting the k-th maximum of $R_1, \ldots, R_n$ (in decreasing order i.e. $M_n^{(1)}$ denotes the maximum).

The sequence $\left({}^{(r)}S_n\right)_{n \geq 1}$ is known as

- *lightly trimmed sum process* if r is a fixed integer.
- *moderately trimmed sum process* if $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$
- *heavily trimmed sum process* if $r_n/n \to c \in (0, 1)$ as $n \to \infty$.

Let r be a fixed integer. We are interested in studying the almost sure convergence of

$$\frac{^{(r)}S_n}{n \log n}.$$

Theorem (Athreya and Athreya (2021))

With probability 1,

$$\lim_{n\to\infty} \frac{S_n - M_n^{(1)}}{n \log n} = 1$$

where S_n represents the partial sum of Lüroth random variables.

Question: Can we obtained convergence results for any trimmed generalized Oppenheim expansion?

- Consider a strictly increasing sequence $\Lambda = (\lambda_j)_{j \in \mathbb{N}}$ tending to $+\infty$ with $\lambda_j \geq 1$ for every $j \geq 1$ and $\lambda_0 = 0$.
- For $u \in [1, +\infty)$ let j_u be the only integer such that $\lambda_{j_u - 1} < u \leq \lambda_{j_u}$ (i.e. λ_{j_u} is the minimum element in Λ larger than or equal to u).

Theorem (Giuliano and Hadjikyriakou (2024))

Consider the random variables $(R_n)_{n \geq 1}$ and assume that there exists a sequence Λ such that for every $x \in \Lambda$ and for every n,

$$x \phi_n(B_n) + (x - 1) Q_n \phi_n(B_n)$$

is an integer. For every n, denote $T_n = \lambda_{j_{R_n}}$. Then T_n takes values in Λ, and the sequence $(T_n)_{n \geq 1}$ consists of independent random variables. Moreover the discrete density of T_n is given by the formula

$$F_n\left(\frac{1}{\lambda_{s-1}}\right) - F_n\left(\frac{1}{\lambda_s}\right), \quad s \in \mathbb{N}^*.$$

- The result above is a generalization of Theorem 3 in Galambos (1974), in which $Q_n = 0$, $\Lambda = \mathbb{N}$ and $F_n(x) = F(x) = x$.

- Recall that the notation q_n stands for the sequence of nonnegative numbers such that $q_n(B_1, \ldots, B_n) = Y_n$. Consider positive integers $a_1, \ldots, a_p$ and assume that

$$\phi_{kp+j-1} = 1/a_j, \quad \text{for} \quad k \in \mathbb{N}, \quad j = 1, \ldots, p.$$

Define $\kappa = L.C.M.(a_1, \ldots, a_p)$ and $\Lambda = (\kappa n)_{n \geq 1}$ and assume that $q_n \equiv c_n$ where $(c_n)_{n \geq 1}$ is a sequence of positive numbers chosen from the set Λ. Then, for any $x \in \Lambda$,

$$x\phi_n(B_n) + (x - 1)Q_n\phi_n(B_n)$$

is an integer.

Theorem

Consider the random variables $(R_n)_{n\geq 1}$ and assume that there exists a sequence Λ such that for every $x \in \Lambda$ and for every n,

$$x\phi_n(B_n) + (x-1)Q_n\phi_n(B_n)$$

is an integer. Moreover assume the following:

(i)
$$\sup_n(\lambda_{n+1} - \lambda_n) = \ell < +\infty;$$

(ii) $F_n \equiv F$ *for all integers n and there exists a constant $\alpha > 0$ such that*
$$\lim_{t\to 0}\frac{F(t)}{t} = \alpha.$$

Then,

$$\lim_{n\to\infty}\frac{S_n - M_n^{(1)}}{n\log n} = \alpha \qquad P-\text{a.s.}.$$

where $S_n = \sum_{i=1}^{n} R_i$ *and* $M_n^{(1)} = \max\{R_1 \ldots, R_n\}$.

Proof.

Let $T_n = \lambda_{j_{R_n}}$ and define $\tilde{M}_n^{(1)} = \max\{T_1 \dots, T_n\}$. Then,

$$T_n - \ell \le R_n \le T_n \qquad \text{and} \qquad \tilde{M}_n^{(1)} - \ell \le M_n^{(1)} \le \tilde{M}_n^{(1)}.$$

Thus,

$$\frac{\sum_{k=1}^n T_k - \tilde{M}_n^{(1)} - \ell n}{n \log n} \le \frac{S_n - M_n^{(1)}}{n \log n} \le \frac{\sum_{k=1}^n T_k - (\tilde{M}_n^{(1)} - \ell)}{n \log n}.$$

The desired conclusion follows by using a Mori's result (1977) for independent random variables which leads to the conclusion that

$$\frac{\sum_{k=1}^n T_k - \tilde{M}_n^{(1)}}{n \log n} \to \alpha, \quad n \to \infty.$$

□

Remark: This result covers the Lüroth case studied in Athereya and Athereya (2021) but it can also be used to derive the respective convergence for the Engel and Sylvester series.

Theorem (Giuliano and Hadjikyriakou (2024))

Consider the random variables $(R_n)_{n\geq 1}$ *and assume that for the involved distribution functions* $(F_n)_{n\geq 1}$ *the following condition is satisfied:*

$$\sup_{n\geq 1} \limsup_{x\to 0} \frac{F_n(x)}{x} < \infty.$$

Then, for every $r \geq 2$ *and* $p > 2$,

$$\lim_{n\to\infty} \frac{M_n^{(r)}}{n\log n} = 0, \qquad P-\text{a.s.}$$

and

$$\lim_{n\to\infty} \frac{{}^{(r)}S_n}{(n\log n)^p} = 0, \qquad P-\text{a.s.}$$

Remarks:

1. The result is valid for any Oppenheim expansion.
2. There is no assumption for the dependence structure of $(R_n)_{n\geq 1}$.
3. The involved distribution functions may differ.

Corollary (Giuliano and Hadjikyriakou (2024))

Consider the random variables $(R_n)_{n\geq 1}$ and assume that for the involved distribution functions $(F_n)_{n\geq 1}$ the following condition is satisfied:

$$\sup_{n\geq 1} \limsup_{x\to 0} \frac{F_n(x)}{x} < \infty.$$

Then, for every $p > 2$,

$$\lim_{n\to\infty} \frac{S_n - M_n^{(1)}}{(n\log n)^p} = 0, \qquad P - \text{a.s.}$$

Proof The result follows easily by observing that for $r \geq 2$

$$\frac{S_n - M_n^{(1)}}{(n\log n)^p} = \frac{{}^{(r)}S_n}{(n\log n)^p} + \sum_{k=2}^{r} \frac{M_n^{(k)}}{(n\log n)^p}$$

and the convergence to zero is established by applying the last two results.

- Athreya J. S. and Athreya K. B. (2021). Extrema of Luroth Digits and a Zeta Function Limit Relation, *Integers*, **21**, A96.
- Galambos, J., (1974). Further ergodic results on the Oppenheim series, *Quart. J. Math.* 25, **1**, 135–141.
- Giuliano, R., (2018). Convergence results for Oppenheim expansions, *Monatsh. Math*, **187**,3, 509–530.
- Giuliano, R., and Hadjikyriakou, M. (2020). On exact laws of large numbers for Oppenheim expansions with infinite mean, *Journal of Theoretical Probability*, **34**, 1579–1606.
- Giuliano, R., Hadjikyriakou, M., (2023), New asymptotic results for generalized Oppenheim expansions. *Acta Math. Hungar.* **169**, 359–381.
- T. Mori, (1977). Stability for sums of i.i.d. random variables when extreme terms are excluded, *Z.Wahrscheinlichkeitstheorie und Verw. Gebiete* **40**(2), 159–167.

Thank you for your attention