Generating families of continuous univariate distributions

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- Koutras, M. V. and Dafnis, S. (2024). A new family of continuous univariate distributions. Under revision in *Methodology and Computing in Applied Probability.*
- Koutras, M. V. and Dafnis, S. (2024). On the generation of continuous univariate distributions. Submitted for publication.

Generating new families of distributions from existing ones

FAMILY 1

Azzalini-Type Skew-Symmetric

FAMILY 2

Transformation of Random Variable

FAMILY 3

Transformation of Scale

FAMILY 4

Probability Integral Transformation of a Random Variable on [0,1]

Generating new families of distributions from existing ones

FAMILY 1

Azzalini-Type Skew-Symmetric

Define the density of X_A to be

$$f_A(x) = 2W(x)f(x)$$

where f is a pdf and $W(\cdot)$ a function such that

$$W(x) + W(-x) = 1$$

The most familiar special cases take $W(x) = F(\alpha x)$ to be the cdf of a (scaled) symmetric distribution

Azzalini, 1985, *Scand. J. Stat.,* Azzalini with Capitanio, 2014, *book*

Generating new families of distributions from existing ones

FAMILY 2

Transformation of Random Variable

Let $W: R \to R$ be an invertible increasing function. If $X \sim f$, then define $X_T = W(X)$. The density of the distribution of X_T is

$$f_T(x) = \frac{f\left(W^{-1}(x)\right)}{W'\left(W^{-1}(x)\right)}$$

Jones & Pewsey, 2009, *Biometrika*

Generating new families of distributions from existing ones

FAMILY 3

Transformation of Scale

The density of the distribution of X_S is just

$$f_S(x)=2f\left(W^{-1}(x)\right)$$

which is a density if

$$W(x) - W(-x) \;= x$$

Jones, 2014, *Statist. Sinica*

Generating new families of distributions from existing ones

FAMILY 4

Probability Integral Transformation of a Random Variable on [0,1]

Let g be the density of a random variable U in (0,1). Then define $X_U = F^{-1}(U)$ where $F' = f$. The density of the distribution of X_U is

$$f_U(x)=f(x)g\big(F(x)\big)$$

An easy task, but .....

It is **one of the easiest things in statistics** to invent new univariate distributions; after all, any non-negative integrable function is the core of a density function. The ongoing challenge is to extract from the overwhelming plethora of possibilities those relatively few with the **best and most appropriate properties that are of real potential value in practical applications**.

Jones, M. C. (2015). On families of distributions with shape parameters. International Statistical Review 83, 2, 175-192.

Motivation

For many classical continuous univariate distributions, there exists a monotone transformation $g(\cdot)$, of their cumulative density function (cdf) $F(x)$, so that

$$g(F(x)) = h(x;\boldsymbol{\theta}),$$

where

- $g(x)$ does not involve any of the distributions parameters.
- $h(x;\boldsymbol{\theta})$ contains the parameters of the distribution.

Well known distributions

$g(F(x)) = h(x; \boldsymbol{\theta})$

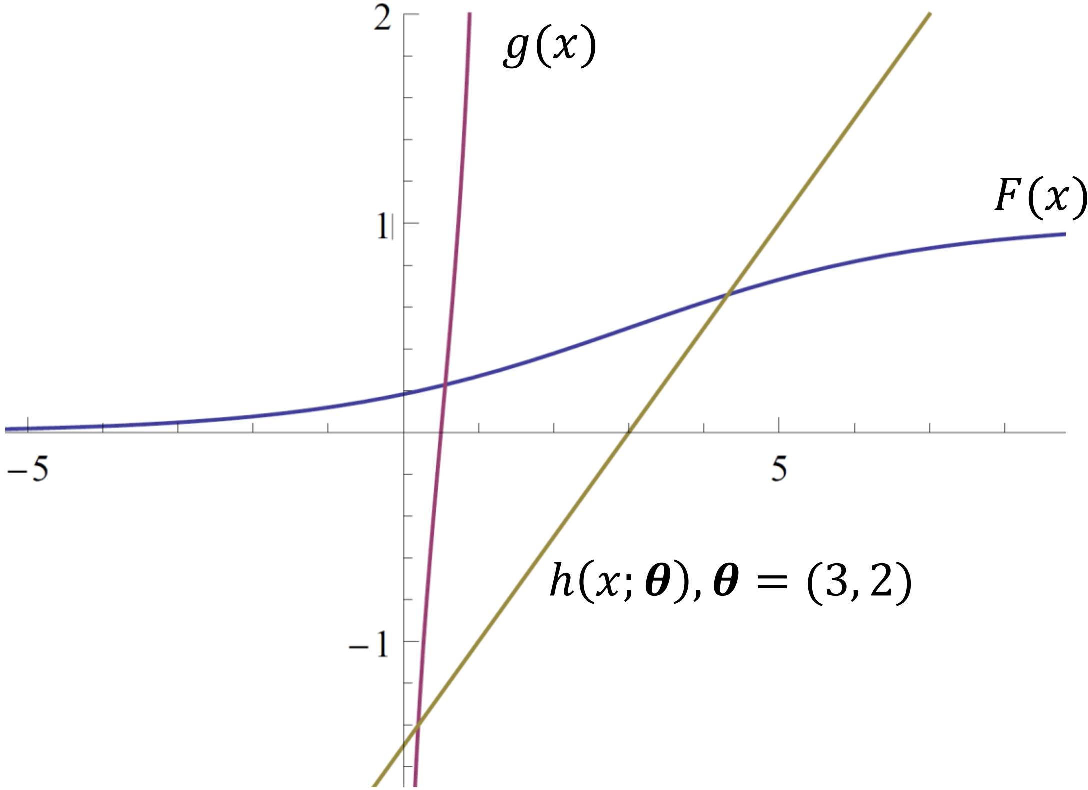
Distribution	$F(x)$	$\boldsymbol{\theta}$	Transformation	$g(x)$	$h(x;\boldsymbol{\theta})$
Exponential	$1-e^{-\lambda x}$	λ	$-\ln(1-F)=\lambda x$	$-\ln(1-x)$	λx
Weibull	$1-e^{-\lambda x^{\alpha}}$	(α,λ)	$-\ln(1-F)=\lambda x^{\alpha}$	$-\ln(1-x)$	λx^{α}
Pareto known α	$1-(\frac{\lambda}{x+\lambda})^{\alpha}$	λ	$(\frac{1}{1-F})^{\frac{1}{\alpha}}=\frac{x}{\lambda}+1$	$(\frac{1}{1-x})^{\frac{1}{\alpha}}$	$\frac{x}{\lambda}+1$
Gompertz	$1-e^{-\alpha(e^{\lambda x}-1)}$	(α,λ)	$-\ln(1-F)=\alpha(e^{\lambda x}-1)$	$-\ln(1-x)$	$\alpha(e^{\lambda x}-1)$
Dagum known p	$\left(1+(\frac{x}{\lambda})^{-\alpha}\right)^{-p}$	(α,λ)	$(F^{-1/p}-1)^{-1}=(\frac{x}{\lambda})^{\alpha}$	$\frac{1}{-1+x^{-1/p}}$	$(\frac{x}{\lambda})^{\alpha}$
Exponential-logarithmic known p	$1-\frac{\ln(1-(1-p)e^{-\lambda x})}{\ln p}$	λ	$-\ln(\frac{1-p^{1-F}}{1-p})=\lambda x$	$-\ln(\frac{1-p^{1-x}}{1-p})$	λx
Log-logistic	$\frac{x^{\alpha}}{\lambda^{\alpha}+x^{\alpha}}$	α	$\ln(\frac{F}{1-F})=\alpha\ln(\frac{x}{\lambda})$	$\ln(\frac{x}{1-x})$	$\alpha\ln(\frac{x}{\lambda})$
Burr known α	$1-(\frac{1}{1+(\frac{x}{\lambda})^{\beta}})^{\alpha}$	β,λ	$(\frac{1}{1-F})^{\frac{1}{\alpha}}=(\frac{x}{\lambda})^{\beta}+1$	$(\frac{1}{1-x})^{\frac{1}{\alpha}}$	$(\frac{x}{\lambda})^{\beta}+1$

Example: Logistic Distribution

$g(F(x)) = h(x;\boldsymbol{\theta})$

$$F(x) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}}, \qquad x \in \mathcal{R}$$

- Transformation: $\ln\left(\frac{F}{1-F}\right) = \frac{x-\mu}{\sigma}$
- Generator: $g(x) = \ln\left(\frac{x}{1-x}\right)$
- Parametric part: $h(x;\boldsymbol{\theta}) = \frac{x-\mu}{\sigma}$



The New Family: Definition

$g(F(x)) = h(x; \boldsymbol{\theta})$

We shall say that a distribution with support $(-\infty, \infty)$ and cdf F belongs to the class $D_g(h)$ (notation: $F \in D_g(h)$) if

$$F(x) = g^{-1}(h(x))$$

where

C1. $g\colon (0,1) \to \mathcal{R}$ is strictly increasing.

C2. $h\colon (-\infty, \infty) \to \mathcal{R}$ is increasing.

C3. g, h differentiable.

C4. $\lim_{x\to 1^-} g(x) = \infty$ and $\lim_{x\to 0^+} g(x) = l$ ($l \in \mathcal{R}$ or $l = -\infty$).

C5. $\lim_{x\to\infty} h(x) = \infty$ and $\lim_{x\to-\infty} h(x) = l$.

Aging Properties of $D_g(h)$ — Proposition 1a

Let $F \in D_g(h)$, $\Delta \subseteq \mathcal{R}$ and $Q(\cdot)$ the function defined by

$$Q(x) = g'(x)(1-x), \quad 0 < x < 1.$$

.... best and most appropriate properties that are of real potential value in practical applications

If

- h' is decreasing in Δ (h concave) and
- Q is increasing in $F(\Delta) \subseteq (0,1)$

then F has a decreasing failure rate ($F \in DFR$) in Δ.

The failure rate of $D_g(h)$ – sketch of Proposition's proof

Let $F \in D_g(h)$, $\Delta \subseteq \mathcal{R}$ and $Q(\cdot)$ the function defined by

$$Q(x) = g'(x)(1-x), \quad 0 < x < 1.$$

If

- h' is decreasing in Δ (h concave) and
- Q is increasing in $F(\Delta) \subseteq (0,1)$

then F has a decreasing failure rate ($F \in DFR$) in Δ.

$$r(x) = \frac{h'(x)\searrow}{Q\nearrow(F\nearrow(x))}$$

Aging Properties of $D_g(h)$ — Proposition 1b

Let $\Delta \subseteq \mathcal{R}$. If
- h' is decreasing in Δ (h concave) and
- Q is increasing in $F(\Delta) \subseteq (0,1)$

then $F \in D_g(h)$ has a decreasing failure rate $(F \in DFR)$ in Δ.

Let $\Delta \subseteq \mathcal{R}$. If
- h' is increasing in Δ (h convex) and
- Q is decreasing in $F(\Delta) \subseteq (0,1)$

then $F \in D_g(h)$ has an increasing failure rate $(F \in IFR)$ in Δ.

Aging Properties of $D_g(h)$ — Proposition 1c

Let $\Delta \subseteq \mathcal{R}$ and assume that h' is constant in Δ (i.e. the parametric part h is linear in x in Δ).

- If Q increasing in $F(\Delta) \subseteq (0,1)$, then $F \in D_g(h)$ is DFR in Δ.
- If Q decreasing in $F(\Delta) \subseteq (0,1)$, then $F \in D_g(h)$ is IFR in Δ.
- If Q constant in $F(\Delta) \subseteq (0,1)$, then $F \in D_g(h)$ has a constant failure rate in Δ.

$$r(x) = \frac{h'(x)}{Q(F(x))} = \frac{\text{constant}}{Q(F(x))}$$

Aging Properties of classical distributions exploiting Proposition 1c

For many classical distributions we have

$$h(x) = cx + d, \quad c > 0.$$

Therefore, the study for the aging properties of the classical distributions can be conferred from the monotonicity properties of the function

$$Q(x) = g'(x)(1 - x), 0 < x < 1.$$

Aging Properties of classical distributions exploiting Proposition 1c

	Distribution	$F(x)$	$g(x)$	$h(x;\theta)$
1	Logistic	$\frac{1}{1+e^{-\frac{x-\mu}{\sigma}}}$	$ln\left(\frac{x}{1-x}\right)$	$\frac{x-\mu}{\sigma}$
2	Gumbel	$\exp(-e^{-\frac{x-\mu}{\sigma}})$	$-ln(-lnx)$	$\frac{x-\mu}{\sigma}$
3	Cauchy	$\frac{1}{2}+\frac{1}{\pi}\arctan\left(-\frac{x-\mu}{\sigma}\right)$	$\tan\left(\pi x-\frac{\pi}{2}\right)$	$\frac{x-\mu}{\sigma}$

$$Q(x)=g'(x)(1-x)$$

Proposition 1c: Application for the Logistic Distribution

$$g(x) = \ln\left(\frac{x}{1-x}\right), \qquad h(x;\boldsymbol{\theta}) = \frac{x-\mu}{\sigma}$$

$$Q(x) = g'(x)(1-x) = \frac{1}{x}$$

$$Q'(x) = -\frac{1}{x^2} < 0$$

Q is decreasing in $F((-\infty,\infty)) = (0,1)$

$\Rightarrow$ F is IFR in $(-\infty,\infty)$

Let $\Delta \subseteq \mathcal{R}$ and assume that h' is constant in Δ.

- If Q increasing in $F(\Delta) \subseteq (0,1)$, then $F \in IFR$ in Δ.
- If Q decreasing in $F(\Delta) \subseteq (0,1)$, then $F \in IFR$ in Δ.
- If Q constant in $F(\Delta) \subseteq (0,1)$, then F has a constant failure rate in Δ.

Probability Bounds for $D_g(h)$

Proposition 3. If $F \in D_g(h)$ and c is a positive lower bound for the function g', i.e.

$$g'(x) \geq c, \qquad \forall\, x \in (0,1)$$

then

$$P(x_1 < X < x_2) \leq \frac{h(x_2;\boldsymbol{\theta}) - h(x_1;\boldsymbol{\theta})}{c},$$

for every $x_1, \; x_2$ with $x_1 < x_2$.

Unimodality of $D_g(h)$

Proposition 2. Let $F \in D_g(h)$ and $w(\cdot)$ the function defined in $\Delta \subseteq \mathcal{R}$ by the formula

$$w(x) = \frac{1}{g'(x)}.$$

.... best and most appropriate properties that are of real potential value in practical applications

If

- w is concave in $F(\Delta)$
- h' is logconcave in Δ
- both h and g are concave functions or both h and g are convex functions in Δ and $F(\Delta) \subseteq (0,1)$, respectively,

then $F \in D_g(h)$ is unimodal and IFR in Δ.

A gallery for h functions to generate distributions in $D_g(h)$

- **Linear**

  $h(x) = \alpha x, \quad \alpha > 0$

- **Power**

  $h(x) = \alpha x^b, \quad \alpha > 0$ and $b \geq 1$ odd

- **Exponential**

  $h(x) = ae^{bx} \; \alpha, b > 0$

- **Exponential – Logarithmic**

  $h(x) = ln\left(ln(a + be^{cx})\right), b, c > 0$ and $a \geq 1$

- **Power - Logarithmic**

  $h(x) = a\, ln(1 + bx^c) \;\; a > 0, b > 0$ and $c \geq 1$ odd

C2. $h: (-\infty, \infty) \to \mathcal{R}$ is increasing.

C3. h differentiable.

C5. $\lim_{x \to \infty} h(x) = \infty$ and $\lim_{x \to -\infty} h(x) = l$ ($l \in \mathcal{R}$ or $l = -\infty$).

Generation of new distributions in $D_g(h)$:linear combinations

Proposition 4. Let

$$F_1 \in D_g(h_1), \qquad F_2 \in D_g(h_2)$$

and introduce the function h_3 defined by

$$h_3 = b_1 h_1 + b_2 h_2,$$

with $b_1,\ b_2 > 0$. Then $F_3 = g^{-1} \circ h_3 \in D_g(h_3)$ if

- $\lim_{x \to 0^+} g(x) = l \in \mathcal{R}$ and $b_1 + b_2 = 1$ or
- $\lim_{x \to 0^+} g(x) = -\infty$ or
- $\lim_{x \to 0^+} g(x) = 0.$

Generation of new distributions in $D_g(h)$: multiplication

Proposition 5. Let

$$F_1 \in D_g(h_1), \qquad F_2 \in D_g(h_2)$$

and introduce the function h_3 defined by

$$h_3 = h_1 h_2.$$

If $\lim_{x \to 0^+} g(x) = l \in \mathcal{R}$ and $l = 0$ or $l = 1$, then

$$F_3 = g^{-1} \circ h_3 \in D_g(h_3).$$

Generation of new distributions in $D_g(h)$: composition

Proposition 6. Let

$$F_1 \in D_g(h_1), \qquad F_2 \in D_g(h_2)$$

and introduce the function h_3 defined by

$$h_3 = h_1 \circ h_2 .$$

If

$$\lim_{x \to 0^+} g(x) = l \in \mathcal{R} \text{ and } h_1(l) = l,$$

then

$$F_3 = g^{-1} \circ h_3 \in D_g(h_3).$$

Generation of new distributions in $D_g(h)$:transforming the generator

Proposition 4.1. *Let* $F \in D_g^+(h)$ *and* $g_0 : (0,1) \to \mathbb{R}$ *a strictly increasing and differentiable function such that*

$$lim_{x \to 0^+} g_0(x) = 0 \ \ and \ \ lim_{x \to 1^-} g_0(x) = 1.$$

Then, the function $g_1 = g \circ g_0$ *is a valid generator leading to the cdf*

$$F^* = {g_1}^{-1} \circ h \in D_{g_1}^+(h).$$

Generation of new distributions in $D_g(h)$:transforming the generator

	$g_0(x)$	${g_0}^{-1}(x)$	Parameters	Comments
1	$(-ln(1-x(1-e^{-1})))^{\frac{1}{\alpha}}$	$\frac{e^{-x^{\alpha}}-1}{e^{-1}-1}$	$\alpha > 0$	Khalil et al. (2021)
2	$\frac{(1-\theta)x}{x-\theta}$	$\frac{\theta x}{\theta-1+x}$	$\theta > 0$	Ahmad et al. (2022)
3	$\frac{ln((\theta-1)x+1)}{ln\theta}$	$\frac{\theta^{x}-1}{\theta-1}$	$0 < \theta \neq 1$	Mahdavi and Kundu (2017)
4	$\left(1-\frac{\theta}{ln(1-x^{\frac{1}{\alpha}})}\right)^{-1}$	$(1-e^{-\theta\frac{x}{1-x}})^{\alpha}$	$\theta > 0, \alpha > 0$	Barati and Rashidi (2022)
5	x^{α}	$x^{\frac{1}{\alpha}}$	$\alpha > 0$	Special case of the $Beta(\alpha,\beta)$ distribution with $\beta = 1$
6	$1-(1-x)^{\beta}$	$1-(1-x)^{\frac{1}{\beta}}$	$\beta > 0$	Special case of the $Beta(\alpha,\beta)$ distribution with $\alpha = 1$
7	$1-(1-x^{\alpha})^{\beta}$	$(1-(1-x)^{\frac{1}{\beta}})^{\frac{1}{\alpha}}$	$\alpha,\beta > 0$	Kumaraswamy Distribution, see e.g. Jones (2009)
8	$\frac{1}{1-lnx}$	$e^{1-\frac{1}{x}}$	–	

Properties of the transformed family

Proposition 5.4. *Let* $F \in D_g^+(h)$, $g_0 : (0,1) \to \mathbb{R}$ *a transformation function and* $F^* \in D_{g,g_0}^+(h)$ *the transformed* D_g^+ *– family. Let also* $s(x)$ *denote the function*

$$s(x) = \frac{g_0'(x)(1-x)}{1-g_0(x)} = -(1-x)(ln(1-g_0(x)))', \ 0 < x < 1.$$

a. *If* F *is IFR on* $\Delta \subseteq (0,\infty)$ *and* $s(x)$ *is decreasing on* $F(\Delta)$ *then* F^* *is IFR on* Δ.

b. *If* F *is DFR on* $\Delta \subseteq (0,\infty)$ *and* $s(x)$ *is increasing on* $F(\Delta)$, *then* F^* *is DFR on* Δ.

Properties of the transformed family

Proposition 5.8. *Let* $F \in D_g^+(h)$, $g_0 : (0,1) \to \mathbb{R}$ *a transformation function with*

$$\lim_{x\to 1^-} g_0'(x) \neq 0$$

and $F^* \in D_{g,g_0}^+(h)$ *the corresponding transformed* D_g^+ *– family.*

a. *If F is heavy tailed, then* F^* *is heavy tailed.*

b. *If F is not heavy tailed, then* F^* *is not heavy tailed.*

Further research

- Additional closure properties (when combining several parametric parts)
- Study of fitting performance in
  - experimental data (hydration heat, antibacterial activity)
  - collections of big data (social networking data)

Thank you for your attention!