# Domino tilings of generalised Aztec triangles

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# Di Francesco's determinant

In 2021, in the context of counting certain configurations in the 20-vertex model, Di Francesco came up with the following conjecture:

**Conjecture**

*For all positive integers n, we have*

$$\det_{0\le i,j\le n-1}\left(2^i\binom{i+2j+1}{2j+1}+\binom{-i+2j+1}{2j+1}\right) = 2^{\binom{n}{2}+1}\prod_{i=0}^{n-1}\frac{(4i+2)!}{(n+2i+1)!}.$$

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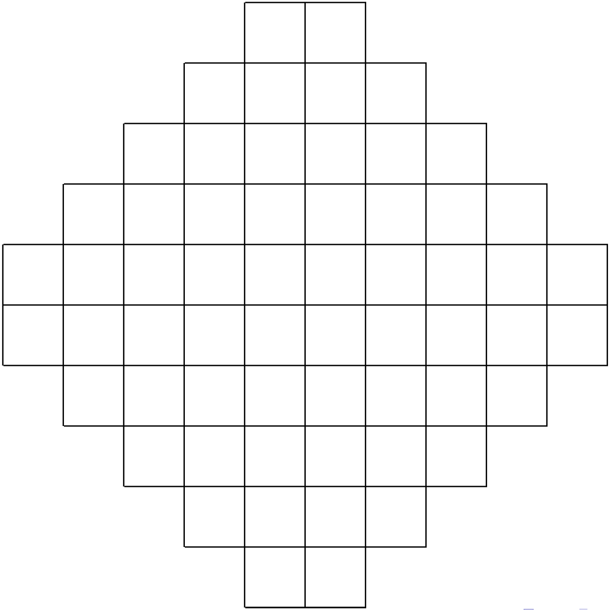
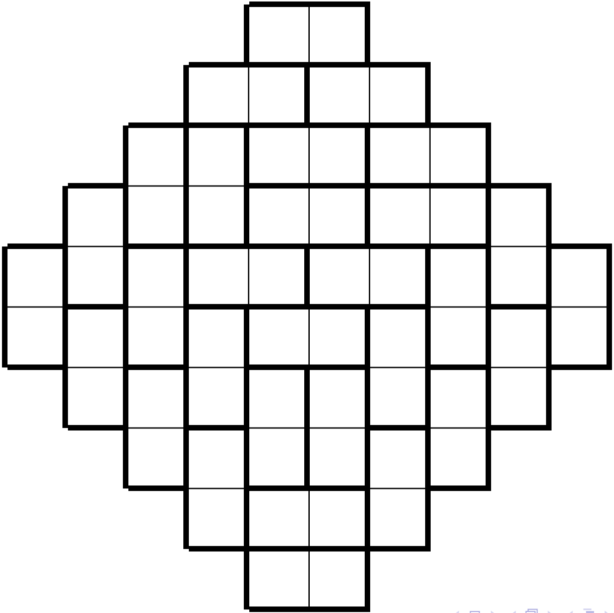
More precisely, Di Francesco observed that the number of domino tilings of certain regions that he called Aztec triangles is the same as the number of these 20-vertex configurations. He showed that the number of domino tilings is given by one half of the above determinant.

# The Aztec diamond



# Domino tilings of the Aztec diamond



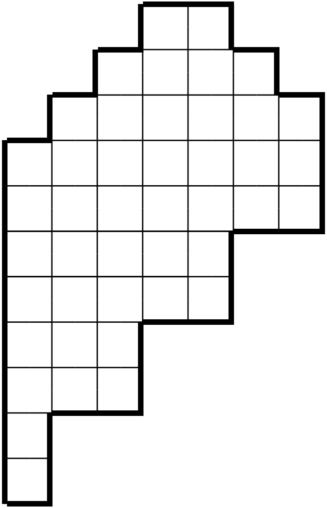
# The Aztec diamond theorem

**Theorem (Elkies, Kuperberg, Larsen, Propp 1992)**

*The number of domino tilings of the Aztec diamond of size $n$ is*

$$2^{\binom{n+1}{2}}.$$

# The Aztec triangle of Di Francesco



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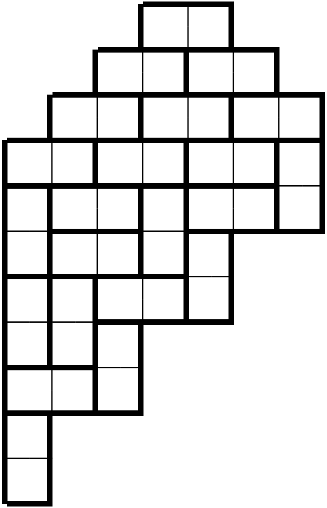

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Christoph Koutschan proved Di Francesco's determinant evaluation using Zeilberger's holonomic Ansatz (and heavy computer calculations).

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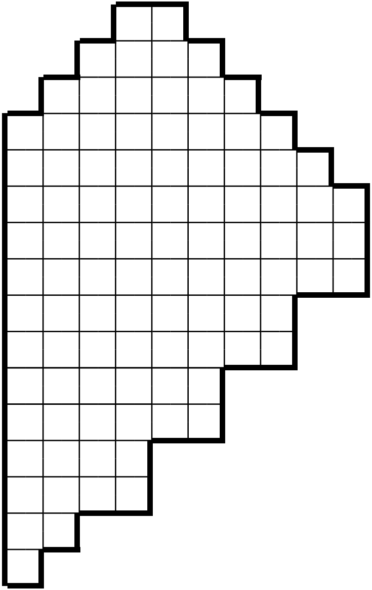
Christoph Koutschan proved Di Francesco's determinant evaluation using Zeilberger's holonomic Ansatz (and heavy computer calculations).

**Theorem (Di Francesco + Koutschan)**

*The number of domino tilings of the Aztec triangle of size n is*

$$2^{\binom{n}{2}}\prod_{i=0}^{n-1}\frac{(4i+2)!}{(n+2i+1)!}.$$

# A generalised Aztec triangle



(Sylvie Corteel and Frederick Huang)

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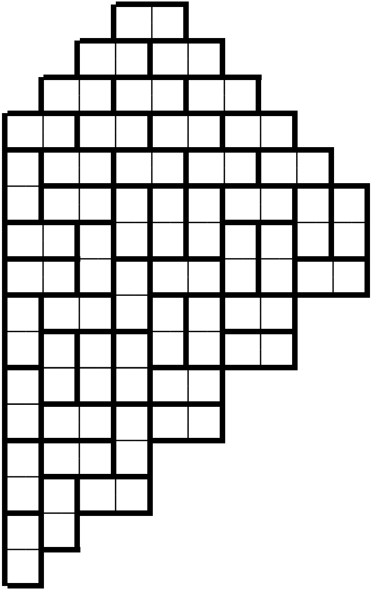

(Sylvie Corteel and Frederick Huang)

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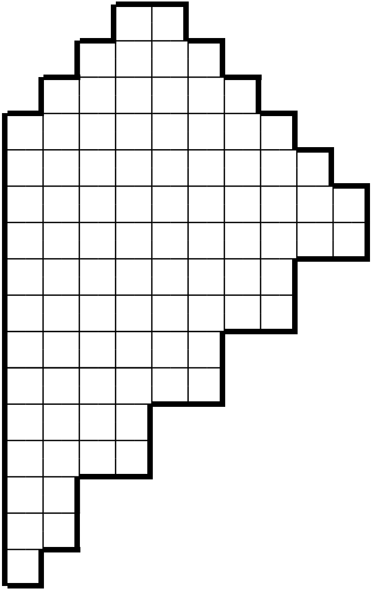
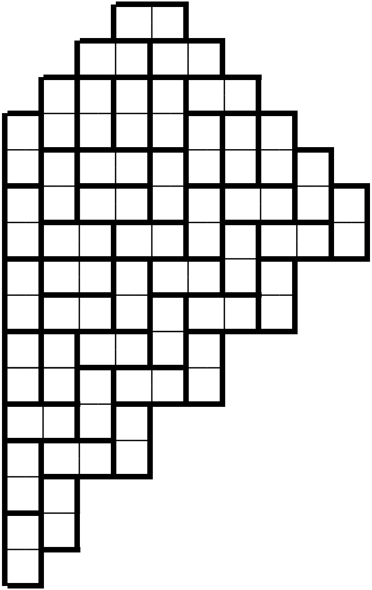


(Sylvie Corteel and Frederick Huang)



(SYLVIE CORTEEL AND FREDERICK HUANG)

# Enumeration of generalised Aztec triangles

**Conjecture (Corteel, Huang)**

*The number of domino tilings of the $(n,k)$-Aztec triangle of type I is*

$$\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

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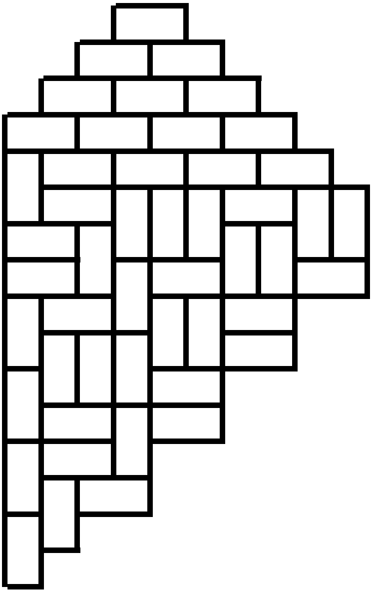
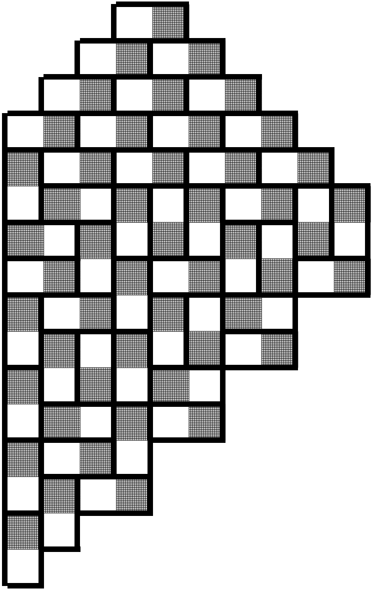
**Conjecture (Corteel, Huang)**

*The number of domino tilings of the $(n, k)$-Aztec triangle of type II is*

$$\prod_{i\geq 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

# Generalised Aztec triangles and non-intersecting paths





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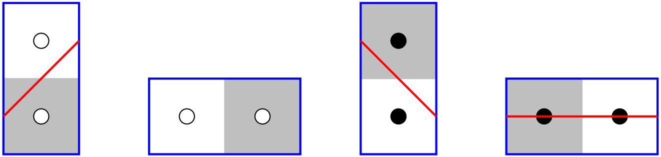

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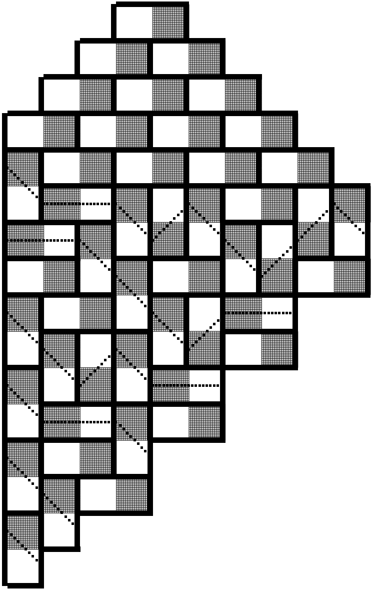

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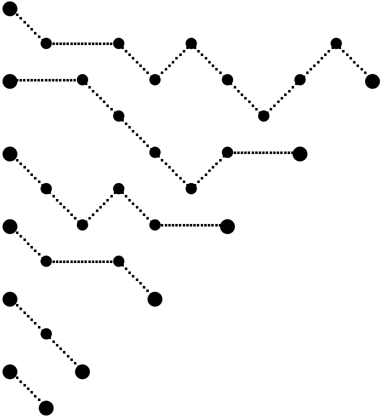

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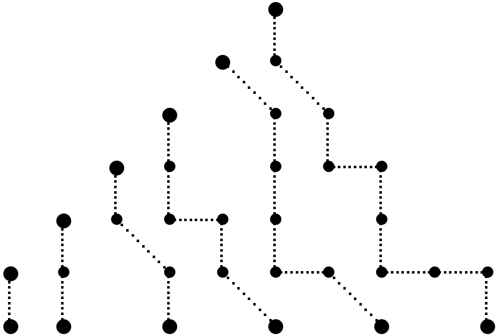

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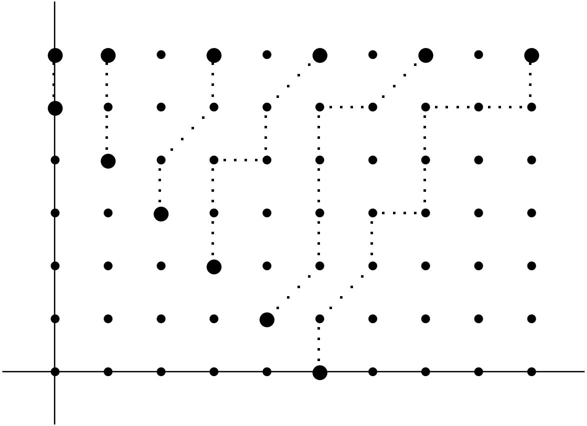

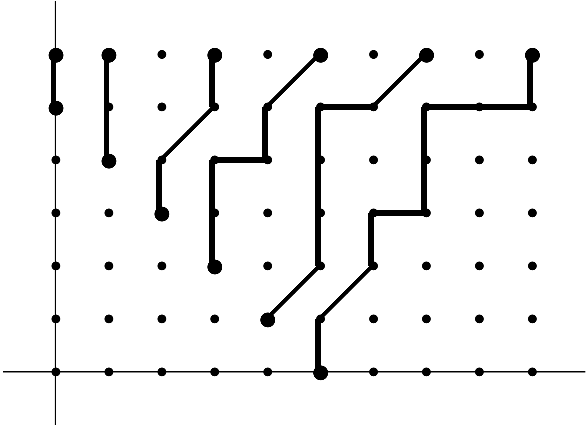

**Theorem (Karlin–McGregor, Lindström, Gessel–Viennot, Fisher, John–Sachs, Gronau–Just–Schade–Scheffler–Wojciechowski)**

*Let $G$ be an acyclic, directed graph, and let $A_1, A_2, \ldots, A_r$ and $E_1, E_2, \ldots, E_r$ be vertices in the graph with the property that, for $i < j$ and $k < l$, any (directed) path from $A_i$ to $E_l$ intersects with any path from $A_j$ to $E_k$. Then the number of families $(P_1, P_2, \ldots, P_r)$ of non-intersecting (directed) paths, where the $i$-th path $P_i$ runs from $A_i$ to $E_i$, $i = 1, 2, \ldots, r$, is given by*

$$\det_{1\le i,j\le r}\left(|\mathcal{P}(A_j \to E_i)|\right),$$

*where $\mathcal{P}(A \to E)$ denotes the set of paths from $A$ to $E$.*

Hence:
*The number of domino tilings of the* $(n,k)$*-Aztec triangle of type I equals* $\det D_1(k;n)$*, where*

$$D_1(k;n) = \left(D(2j-i, i+n-k-1)\right)_{1\le i,j\le k},$$

with $D(m,n)$ a Delannoy number, i.e., the number of paths from $(0,0)$ to $(m,n)$ consisting of steps $(1,0)$, $(0,1)$, and $(1,1)$.

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*Furthermore, the number of domino tilings of the* $(n, k)$*-Aztec triangle of type II equals* $\det D_2(k; n)$, *where*

$$D_2(k; n) = \left(D(2j - i, i + n - k - 1) + D(2j - i - 1, i + n - k - 1)\right)_{1 \le i,j \le k}.$$

# Two determinant evaluations

We need to show:

$$\det\left(D(2j-i,i+n-k-1)\right)_{1\le i,j\le k}$$
$$=\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Big/\prod_{i=1}^{k-1}(2i+1)^{k-i}\,,$$

and also:

$$\det\left(D(2j-i,i+n-k-1)+D(2j-i-1,i+n-k-1)\right)_{1\le i,j\le k}$$
$$=\prod_{i\ge 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Big/\prod_{i=1}^{k-1}(2i+1)^{k-i}\,.$$

# The Delannoy numbers

We have

$$D(m,n) = \langle u^m v^n \rangle \frac{1}{1-u-v-uv} = \sum_{\ell=0}^{m} \binom{m}{\ell}\binom{n}{\ell} 2^\ell = \sum_{\ell=0}^{m} \binom{m+n-\ell}{m-\ell,\, n-\ell,\, \ell}.$$

In particular, $D(m,n)$ is a polynomial in $n$ of degree $m$.

# Determinant evaluations: Identification of factors

**A short proof of the Vandermonde determinant evaluation**

$$\det\left(X_i^{j-1}\right)_{1\le i,j\le n}=\prod_{1\le i<j\le n}(X_j-X_i).$$

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PROOF.

- If $X_{i_1}=X_{i_2}$ with $i_1\ne i_2$, then the determinant vanishes. Hence,

$$\prod_{1\le i<j\le n}(X_j-X_i) \text{ divides } \det_{1\le i,j\le n}\left(X_i^{j-1}\right)$$

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- The degree of the product is $\binom{n}{2}$.
  The degree of the determinant is at most $\binom{n}{2}$.
  Consequently,

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right)=\text{const.}\times\prod_{1\le i<j\le n}(X_j-X_i).$$

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  Consequently,

$$\det_{1\le i,j\le n}\left(X_i^{j-1}\right)=\text{const.}\times\prod_{1\le i<j\le n}(X_j-X_i).$$

- One can compute the constant by comparing coefficients of $X_1^0X_2^1\cdots X_n^{n-1}$ on both sides.

# Determinant evaluations: Identification of factors

What are the essential steps?

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(1) Identification of factors
(2) Comparison of degrees
(3) Evaluation of the constant

**Objection:** This works because there are so many (to be precise: $n$) variables at our disposal.

*What, if there is, say, only* one *variable* $\mu$*, and you want to prove that* $(\mu + a)^E$ *divides the determinant?*

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*What, if there is, say, only* one *variable $\mu$, and you want to prove that $(\mu + a)^E$ divides the determinant?*

**Important fact:**

> *For proving that $(\mu + a)^E$ divides the determinant, we put $\mu = -a$ in the matrix and find $E$ linearly independent vectors in the kernel of the matrix.*

# Our determinant

Here is our wanted determinant evaluation, rewritten:

$$\det D_1(k;n) = \det\big(D(k-2i+j, n-j-1)\big)_{0\le i,j\le k-1}$$

$$= 2^{k^2}\prod_{i=1}^{k}\frac{1}{(i)_i}\prod_{s=0}^{k-2}(n-s-1)^{\min\{\lfloor(s+1+\chi(k\text{ even}))/2\rfloor,\lfloor(k-s)/2\rfloor\}}$$

$$\cdot\prod_{s=0}^{k-1}(n-s-\tfrac{1}{2})^{\min\{\lfloor(s+1+\chi(k\text{ odd}))/2\rfloor,\lfloor(k-s+1)/2\rfloor\}}$$

$$\cdot\prod_{s=0}^{k-2}(n+k-s-1)^{\min\{\lfloor(s+2)/2\rfloor,\lfloor(k-s-\chi(k\text{ odd}))/2\rfloor\}}$$

$$\cdot\prod_{s=1}^{k-2}(n+k-s-\tfrac{1}{2})^{\min\{\lfloor(s+1)/2\rfloor,\lfloor(k-s-\chi(k\text{ even}))/2\rfloor\}}.$$

Here, $\chi(\mathcal{S})=1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S})=0$ otherwise.

# Our determinant

STEP 1: *The term* $(n-s-1)^{\min\{\lfloor (s+1+\chi(k \text{ even}))/2\rfloor,\lfloor (k-s)/2\rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

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*Claim*: For $0 \le a \le s$, $k \ge 2s-a+2$, and $k \equiv a \pmod 2$, we have

$$\sum_{j=a}^{2s-a+1} \binom{2s-2a+1}{j-a} \cdot \big(\text{column } j \text{ of } D_1(k;s+1)\big) = 0.$$

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Concentrate on row $i$ and use one of our formulae for the Delannoy numbers:

$$\sum_{j=a}^{2s-a+1}\binom{2s-2a+1}{j-a}\sum_{\ell=0}^{n-2i+j}\binom{n-2i+s-\ell}{n-2i+j-\ell,\,s-j-\ell,\,\ell}.$$

The sum over $j$ van be simplified using the Chu–Vandermonde convolution formula.

# Our determinant

STEP 2: *The term* $(n - s - \frac{1}{2})^{\min\{\lfloor (s+1+\chi(k \text{ odd}))/2 \rfloor, \lfloor (k-s+1)/2 \rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

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$$\sum_{j=a}^{2s-a}\binom{2s-2a}{j-a}\cdot\left(\text{column } j \text{ of } D_1(k;s+\tfrac{1}{2})\right)=0.$$

# Our determinant

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This is done similarly as in Step 1.

# Our determinant

STEP 3: *The term* $(n+k-s-1)^{\min\{\lfloor (s+2)/2 \rfloor, \lfloor (k-s-\chi(k \text{ odd}))/2 \rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

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*Claim*: For $0 \le 2a \le s$ and $k \ge 2s-2a+2$, we have

$$\sum_{i=a}^{s+1-a} (-1)^i \binom{s+1-2a}{i-a} \cdot \big(\text{row } i \text{ of } D_1(k;-k+s+1)\big)$$

$$-\sum_{i=s+1-a}^{k-1} 2^{2s+2-4a} \binom{i-a-1}{s-2a} \cdot \big(\text{row } i \text{ of } D_1(k;-k+s+1)\big) = 0.$$

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Equivalently:

$$\sum_{i=a}^{s+1-a} (-1)^i \binom{s+1-2a}{i-a} \cdot D(k-2i+j, -k+s-j)$$
$$= \sum_{i=s+1-a}^{k-1} 2^{2s+2-4a} \binom{i-a-1}{s-2a} \cdot D(k-2i+j, -k+s-j).$$

# Our determinant

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This identity can be proved by using the calculus of complex contour integrals.

# Our determinant

STEP 4: *The term* $(n+k-s-\frac{1}{2})^{\min\{\lfloor (s+1)/2\rfloor,\lfloor (k-s-\chi(k\text{ even}))/2\rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

Step 4: *The term* $(n+k-s-\frac{1}{2})^{\min\{\lfloor (s+1)/2\rfloor, \lfloor (k-s-\chi(k \text{ even}))/2\rfloor\}}$ *is a factor of* $\det D_1(k;n)$.

*Claim A*: For $0 \le a < s$ and $k \ge 4s-2a-1$, we have

$$\sum_{i=a}^{k-1} c_1(s-a,i-a)\cdot\big(\text{row } i \text{ of } D_1(k;-k+2s-\tfrac{1}{2})\big)=0.$$

*Claim B*: For $0 \le a < s$ and $k \ge 4s-2a+1$, we have

$$\sum_{i=a}^{k-1} c_2(s-a,i-a)\cdot\big(\text{row } i \text{ of } D_1(k;-k+2s+\tfrac{1}{2})\big)=0.$$

# Our determinant

Here,

$$c_1(s,l) = -\frac{(4l-4s+1)\,(-1)^{s-1}(1-l)_{s-1}\,\left(\frac{1}{2}\right)_s\,\left(\frac{1}{2}\right)_{l-s}}{(2l-4s+1)\,l!\,(s-1)!}$$

$$-(4l-4s+1)\sum_{r=1}^{s}\frac{2^{4r-3}(2r-\frac{1}{2})_{s-r}\,(2r-\frac{1}{2})_{l-s-r}}{(s-r)!\,(l-s-r+1)!}$$

and

$$c_2(s,l) = \frac{(4l-4s-1)\,(-1)^{s-1}(1-l)_{s-1}\,\left(\frac{1}{2}\right)_{s+1}\,\left(\frac{1}{2}\right)_{l-s-1}}{(2l-4s-1)\,l!\,(s-1)!}$$

$$-(4l-4s-1)\sum_{r=1}^{s}\frac{2^{4r-1}(2r+\frac{1}{2})_{s-r}\,(2r+\frac{1}{2})_{l-s-r-1}}{(s-r)!\,(l-s-r)!}.$$

## Our determinant

We consider the proof of the first (triple sum) identity. Concentrating on the $j$-th column, we must show

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Using contour integrals again, the triple sum can be simplified to a double sum.

# Our determinant

Once worked out, we must prove the double sum identity

$$\sum_{i=0}^{k-1}\sum_{\ell=0}^{k-2i}\frac{(4i-4s+1)\,(-1)^{s-1}(1-i)_{s-1}\left(\frac{1}{2}\right)_s\left(\frac{1}{2}\right)_{i-s}}{(2i-4s+1)\,i!\,(s-1)!}$$

$$\cdot\binom{k-2i}{\ell}\binom{-k+2s-\frac{3}{2}}{\ell}2^{\ell}$$

$$+\sum_{r=1}^{s}\sum_{\ell=0}^{k-2r-2s+2}(-1)^k2^{k-2r-2s+3-\ell}\frac{2^{4r-3}(2r-\frac{1}{2})_{s-r}}{(s-r)!}$$

$$\cdot\frac{(k-2r-2s+1)!\,(k+2r-2s-1)!}{\ell!^2\,(k-2r-2s+2-\ell)!\,(k+2r-2s-\ell)!}$$

$$\cdot(-\ell^2+2k+k^2+4r-4r^2-4s-4ks+4s^2)=0,$$

for all integers $k$ and $s$ with $0<s$ and $k\geq 4s-1$.

# Our determinant

How to prove this crazy identity?

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THE SHORT VERSION: Using Christoph Koutschan's *Mathematica* package `HolonomicFunctions`, this is a routine task.

# Our determinant

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Recall:

$$D_1(k;n) = \big(D(k-2i+j, n-j-1)\big)_{0\le i,j\le k-1}.$$

Hence, the degree of det $D_1(k;n)$ is at most

$$\max_{\sigma\in S_k}\left(\sum_{i=0}^{k-1}(k-2i+\sigma(i))\right) = \max_{\sigma\in S_k}\left(k^2-\sum_{i=0}^{k-1} i\right) = \frac{k(k+1)}{2}.$$

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Moreover, it is easy to see directly that the claimed product formula has degree $\binom{k+1}{2}$ as a polynomial in $n$.

# Our determinant

STEP 6: *The coefficient of* $n^{k(k+1)/2}$ *in* $D_1(k;n)$ *is* $2^{k^2}\prod_{i=1}^{k}\frac{1}{(i)_i}$.

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We must compute the determinant of leading coefficients:

$$\det_{0\le i,j\le k-1}\left(\frac{2^{k-2i+j}}{(k-2i+j)!}\right)$$
$$=2^{\binom{k+1}{2}}\prod_{i=0}^{k-1}\frac{1}{(2k-2i-1)!}\det_{0\le i,j\le k-1}\left((k-2i+j+1)_{k-j-1}\right)$$
$$=2^{\binom{k+1}{2}}\prod_{i=0}^{k-1}\frac{1}{(2k-2i-1)!}\det_{0\le i,j\le k-1}\left((-2i)^{k-j-1}\right)$$
$$=2^{\binom{k+1}{2}}\prod_{i=0}^{k-1}\frac{1}{(2k-2i-1)!}\prod_{0\le i<j\le k-1}\left((-2i)-(-2j)\right)$$
$$=2^{\binom{k+1}{2}+\binom{k}{2}}\prod_{i=0}^{k-1}\frac{i!}{(2k-2i-1)!}=2^{k^2}\prod_{i=1}^{k}\frac{1}{(i)_i}.$$

# Our determinant

# The number of domino tilings of the generalised Aztec triangle of type I

**Theorem**

*The number of domino tilings of the $(n, k)$-Aztec triangle of type I is*

$$\prod_{i\geq 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s)\prod_{s=k-2i}^{2k-4i-2}(2n+s)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}\ .$$

# The second determinant

Recall that there was also the determinant of

$$D_2(k;n)$$
$$= \left(D(2j-i,i+n-k-1)+D(2j-i-1,i+n-k-1)\right)_{1\le i,j\le k}.$$

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$$D_2(k;n) = \left(D(2j-i, i+n-k-1) + D(2j-i-1, i+n-k-1)\right)_{1\le i,j\le k}.$$

**Lemma**

*We have*

$$\det D_1(k; n+\tfrac{1}{2}) = \det D_2(k;n).$$

**Proof.**

In fact, we have

$$D_1(k; n+\tfrac{1}{2}) = \left(\frac{(\frac{1}{2})_{j-i}}{(j-i)!}\right)_{0\le i,j\le k-1} \cdot D_2(k;n).$$

□

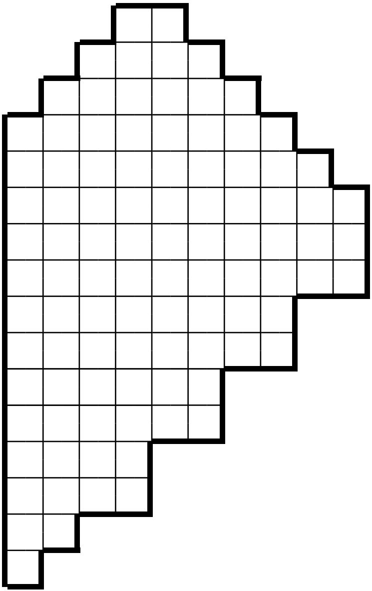
# The number of domino tilings of the generalised Aztec triangle of type II

**Theorem**

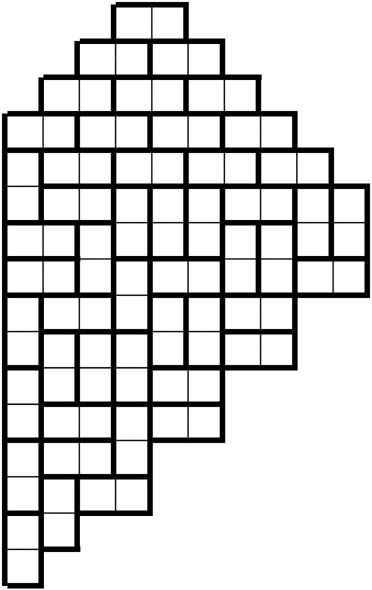
*The number of domino tilings of the* $(n, k)$*-Aztec triangle of type II is*

$$\prod_{i\geq 0}\left(\prod_{s=-2k+4i+1}^{-k+2i}(2n+s+1)\prod_{s=k-2i}^{2k-4i-2}(2n+s+1)\right)\Bigg/\prod_{i=1}^{k-1}(2i+1)^{k-i}.$$

# A generalised Aztec triangle



# A generalised Aztec triangle



# A generalised Aztec triangle