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Part 1: Recursive relations for the moments of the states' sizes of an Homogeneous Markov system (HMS)

Part 2: The continuous time closed Homogeneous Markov system (HMS) as an elastic medium. The 3-d case.

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Contents

Relative research topics

•1. Study of the evolution of the state vector – asymptotic behaviour and rate of convergence of open or closed HMS and NHMS with or without periodicity

•2. Model control

•3 Study of the evolution of the state vector – asymptotic behaviour of the Semi-Markov systems (HSMS NHSMS)

•4. HMS and HSMS as continuous media (fluids, elastic media) – stress tensor – energy of the model

•5. Hidden HMS and HSMS

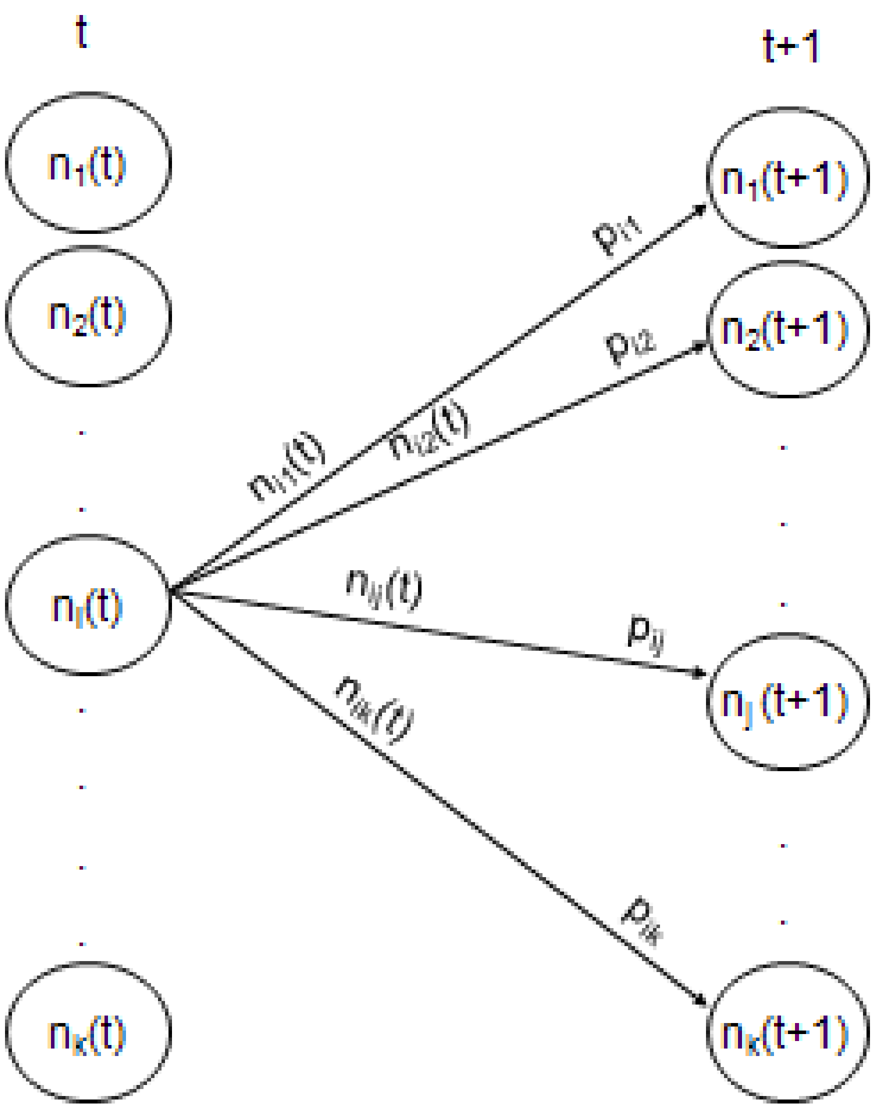
•6. Rewards of HMS and HSMS

•7. Study of (stochastic) matrices involved

•8. Application in many domains

Discrete time closed Homogeneous Markov system (HMS)

- $S = \{1,2,\ldots,n\}$: State space of the HMS
- p_{ij}, $i,j=1,2,\ldots,n$: transition probability of moving from state i to state j
- t, $t>0$: time variable
- $\mathbf{P}=(p_{ij})$, $i=1,2,\ldots,n$, $j=1,2,\ldots,n+1$, the transition matrix
- $x_i(t)$, $i=1,2,\ldots,n$: probability of being in state i at time t
- $\mathbf{x}(t)=(x_1(t), x_2(t),\ldots, x_n(t))^T$: probability state vector of the HMS
- $n_i(t), i = 1, \ldots, k$ r.v. representing the number of members in state i at time t
- $n_{ij}(t), i, j = 1, \ldots, k$ r.v. representing the number of members moving from state i to state j at time period [t,t+1)
- $\boldsymbol{n}(t) = (n_1(t), n_2(t), \ldots., n_k(t))^T$ the state vector of the system



t
t+1
n1(t)
n2(t)
ni(t)
nk(t)
n1(t+1)
n2(t+1)
nj (t+1)
nk(t+1)
ni1(t)
ni2(t)
nij(t)
nik(t)
pi1
pi2
pij
pik

Expected values, variances, covariances of the state sizes of a discrete-time Homogeneous Markov System (HMS)

The following recursive relations have been used to study the behaviour of the state vector.

$$E\left[\mathbf{n}^T(t+1)\right] = E\left[\mathbf{n}^T(t)\right]\mathbf{P},$$

$$\mu^T(t+1) = \mu^T(t)\mathbf{\Pi},$$

where, $$\mu(t) = \{E(n_1(t)), \ldots, E(n_k(t)), cov(n_1(t), n_1(t)), \ldots, cov(n_1(t), n_k(t)), cov(n_2(t), n_1(t)), cov(n_2(t), n_2(t)), \ldots, cov(n_k(t), n_k(t))\}^T,$$

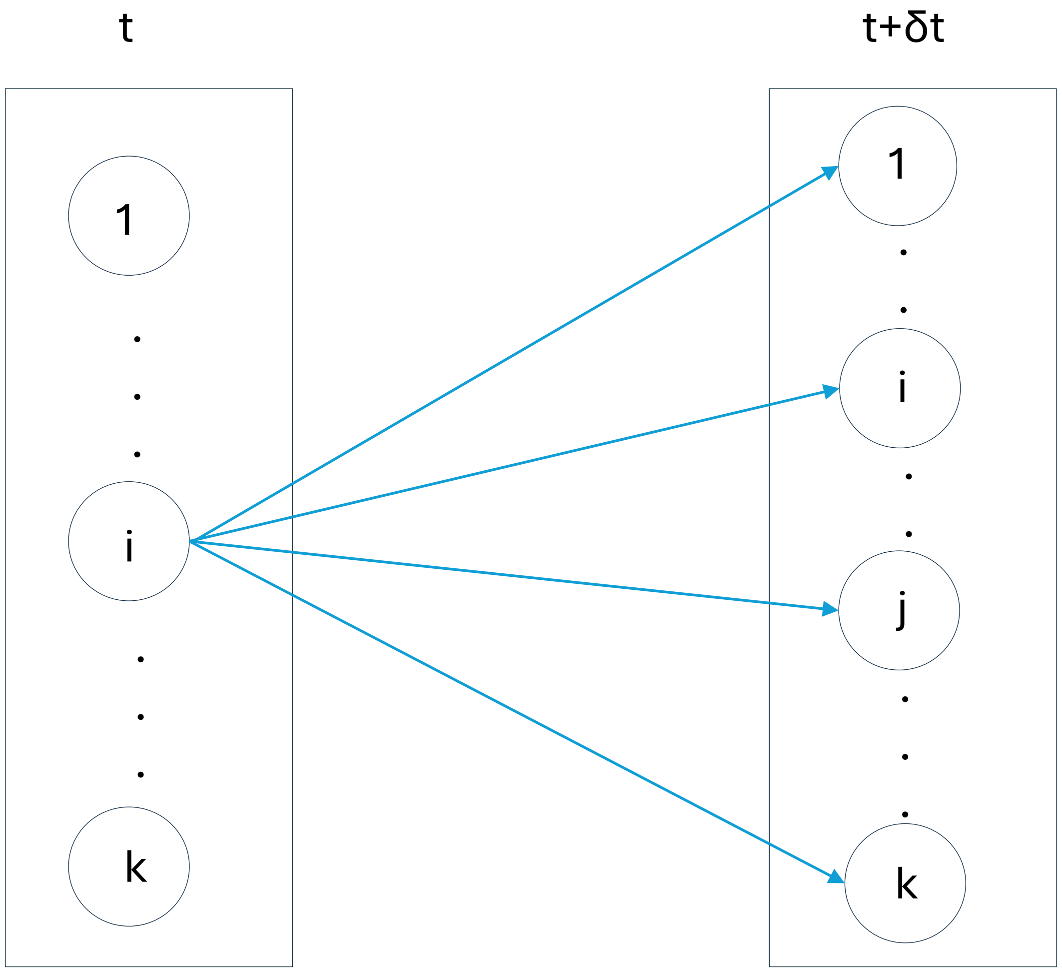
Matrix **Π** with dimensions k(k+1) x k(k+1), is of the form $\begin{bmatrix} \mathbf{P} & \mathbf{X} \\ \mathbf{O} & \mathbf{Y} \end{bmatrix}$,

where **P**=(p_{ij}) is the kxk transition matrix. **O** is the k^2 x k zero matrix. **X** is a k x k^2 matrix with elements of the form $\delta_{jr}p_{ij} - p_{ij}p_{ir}$ (i denotes the row and (j-1)k+r denotes the column of each element) and **Y** is the Kronecker product of **P** and **P** ($\boldsymbol{Y} = \boldsymbol{P} \otimes \boldsymbol{P}$)

Closed continuous- time homogeneous Markov System

- **t:** t>0 time variable
- S={1,2,…,k}: State space of the HMS
- N: number of members in the system
- $\boldsymbol{P}(t, t+\delta t) = (p_{ij}(\mathrm{t}, \mathrm{t}+\delta\mathrm{t}))$, i,j=1,2,…,k is the transition probability matrix for [t, t+δt)
- $\boldsymbol{Q}=(q_{ij})$ i,j=1,2,…,k is the infinitesimal matrix of the transition rates
- $n_{ij}(t, t+\delta t)$, i,j=1,…,k is the number of members moving from state i to state j at time period [t, t+δt)
- $n_i(t)$, i=1,2,..,k is the number of members in state i at time t
- $\boldsymbol{n}(t) = (n_1(t), n_2(t), \dots, n_k(t))^T$ is the system’s state vector at time t

Continuous time Homogeneous Markov system (HMS)



The **transition probabilities** for a time period [t,t+δt) satisfy the equation

$$p_{ij}(t, t+\delta t) = \delta_{ij} + q_{ij}\delta t + o(\delta t),$$

where, δ_{ij} i,j=1,2,..,k is Kronecker's delta, q_{ij} are the transition rates and o(δt) is a quantity for which $\lim_{\delta t \to 0}\left(\frac{o(\delta t)}{\delta t}\right) = 0$

It can be proved that,

$$\frac{d(E[\mathbf{n}^T(t)])}{dt} = E[\mathbf{n}^T(t)]\mathbf{Q},$$

where we denote

$$\frac{d(E[\mathbf{n}^T(t)])}{dt} = \left(\frac{d(E[n_1(t)])}{dt}, \frac{d(E[n_2(t)])}{dt}, \ldots, \frac{d(E[n_k(t)])}{dt}\right).$$

By solving t

$$E\left[\mathbf{n}^T(t)\right] = \mathbf{n}^T(0)e^{\mathbf{Q}t}.$$

Markov models with interactions in the movements

In the case of the models with interactions (not studied here), we assume that if **n**(t) is the system's state vector at time t, then the elements of the transition matrix for the time period [t,t+1) depend on the observed values of **n**(t). Hence, the transition matrix for the time period [t,t+1) is denoted as **P**(**n**(t)). Therefore, the expected size of the system's states at time t+1 given **n**(t), is given by the equation

Therefore,

$$E[\mathbf{n}^T(t+1)/\mathbf{n}^T(t)] = \mathbf{n}^T(t)\mathbf{P}(\mathbf{n}(t)),$$

$$E[\mathbf{n}^T(t+1)] = E[\mathbf{n}^T(t)\mathbf{P}(\mathbf{n}(t))].$$

Conlisk (1976) suggested a deterministic approach, resulting in the replacement of **n**(t) in **P**(**n**(t)) with $E[\boldsymbol{n}^T(\mathrm{t})]$, then

$$E[\mathbf{n}^T(t+1)] \simeq E[\mathbf{n}^T(t)]\mathbf{P}(E[\mathbf{n}^T(t)]).$$

Factorial moments of the states' sizes of a discrete time HMS

Lemma 1.1: Let X be a random variable which is the sum of k independent variables X_i , i=1,2,...,k that follow Binomial distribution with parameters n_i and p_i . For the probability generating function $\Pi_X(z)$ of the random variable X yields,

$$\frac{d^r \Pi_X(z)}{dz} = \sum_{x_1+\ldots+x_k=r} \frac{r!}{x_1!\ldots x_k!} \prod_{i=1}^{k} n_i^{(x_i)} p_i^{x_i} \prod_{i=1}^{k} (1-p_i+p_i z)^{n_i-x_i} ,$$

where the summation is stretched over all x_i=0,1,..,r , i=1,2,...,k such that $x_1 + \cdots + x_k = r$ and

$$n_i^{(x_i)} = n_i(n_i-1)\ldots(n_i-x_i+1), \quad n_i^{(0)} = 1, \text{ για } i = 1,2,\ldots,k.$$

Proposition 1.1: For a closed HMS, with transition matrix **P**=(p_{ij}) $i, j \in S$, r^{th}-order moment $E[n_j^r(t+1)]$ of the random variable. $n_j(t+1), j = 1,2,\dots,k$ are given by the recursive relation:

$$E[n_j^{(r)}(t+1)] = \sum_{x_1+\ldots+x_k=r} \frac{r!}{x_1!x_2!\ldots x_k!} E[n_1^{(x_1)}(t) n_2^{(x_2)}(t) \ldots n_k^{(x_k)}(t)] p_{1j}^{x_1} p_{2j}^{x_2} \ldots p_{kj}^{x_k}.$$

Proposition 1.2: For a closed HMS, with transition matrix **P**=(p_{ij}) $i, j \in S$, the mixed moments of the random variable $n_i(t), i = 1,2,\dots,k$ are given by the recursive relation:

$$E\left[\prod_{i=1}^{k} n_i^{(r_i)}(t+1)\right] = \sum_{x_{11}+\ldots+x_{k1}=r_1} \cdots \sum_{x_{1k}+\ldots+x_{kk}=r_k} \left(\prod_{j=1}^{k} \frac{r_j!}{x_{1j}!\ldots x_{kj}!}\right) \cdot E\left[\prod_{i=1}^{k} n_i^{\left(\sum_{j=1}^{k} x_{ij}\right)}(t)\right] \prod_{j=1}^{k}\prod_{i=1}^{k} p_{ij}^{x_{ij}},$$

where,

$$r_i \in \mathbb{N}, i = 1,2,\dots,k.$$

We define a **vector product** (we use symbol x), that looks like the Kronecker's product.

For example, if $x^T = (x_1, x_2)$ then,

$$\mathbf{x}^T \otimes \mathbf{x}^T = (x_1^2, x_1 x_2, x_2 x_1, x_2^2),$$

and,

$$\mathbf{x}^T \times \mathbf{x}^T = (x_1^{(2)}, x_1^{(1)} x_2^{(1)}, x_2^{(1)} x_1^{(1)}, x_2^{(2)}),$$

That is,

$$\mathbf{x}^T \times \mathbf{x}^T = (x_1(x_1 - 1), x_1 x_2, x_2 x_1, x_2(x_2 - 1)).$$

Theorem 1.1: For a closed, discrete time HMS with transition matrix **P**, it holds that

$$E[\underbrace{\mathbf{n}^T(t+1) \times \ldots \times \mathbf{n}^T(t+1)}_{r}] = E[\underbrace{\mathbf{n}^T(t) \times \ldots \times \mathbf{n}^T(t)}_{r}](\underbrace{\mathbf{P} \otimes \ldots \otimes \mathbf{P}}_{r})$$

and

$$E[\underbrace{\mathbf{n}^T(t) \times \ldots \times \mathbf{n}^T(t)}_{r}] = (\underbrace{\mathbf{n}^T(0) \times \ldots \times \mathbf{n}^T(0)}_{r})(\underbrace{\mathbf{P}^t \otimes \ldots \otimes \mathbf{P}^t}_{r}).$$

Corollary 1.1: If the transition matrix **P** of a closed, discrete time HMS is fully regular , then

$$E[\underbrace{\mathbf{n}^T(\infty) \times \ldots \times \mathbf{n}^T(\infty)}_{r}] = N^{(r)}(\underbrace{\pi^T \otimes \pi^T \otimes \ldots \otimes \pi^T}_{r}),$$

where, $\mathbf{n}^T(\infty) = \lim_{t\to\infty} \mathbf{n}^T(t)$ and N is the size of the system.

Closed HMSs with periodic transition matrix

Let that the transition matrix **P** is periodic, with period d. Then, **P** can be written as,

$$
\mathbf{P} = \begin{array}{c} \\ C_0 \\ C_1 \\ \vdots \\ C_{d-2} \\ C_{d-1} \end{array}
\begin{array}{c} \begin{array}{ccccc} C_0 & C_1 & C_2 & \dots & C_{d-1} \end{array} \\
\begin{pmatrix}
0 & \mathbf{P}_0 & 0 & \dots & 0 \\
0 & 0 & \mathbf{P}_1 & \dots & 0 \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & \mathbf{P}_{d-2} \\
\mathbf{P}_{d-1} & 0 & 0 & \dots & 0
\end{pmatrix} \end{array} \qquad (3.1)
$$

where C_i ,i=0,1,..,d-1 are the cyclical subclasses and $\mathbf{P}_i$,i=0,1,...,d-1 block matrices with dimension $n_{C_i} x\, n_{c_{i+1}}, \mathrm{i} = 0,1, \dots, \mathrm{d} - 1$, respectively, where n_{C_i} is the number of states of class C_i $(C_d \equiv C_0)$

We denote by $n_{C_i}^T(t), i = 0,1,\ldots,d-1$ the vector that contains the sizes of the states that belong to the cyclical subclass $C_{i.}$

Proposition 1.2: The r^{th}-order factorial moments of the sizes of the states of a closed HMS with periodic transition matrix **P** of the form (3.1) are given by the relation:

$$E[\mathbf{n}_{C_{i_1}}^T(t+1)\times\ldots\times\mathbf{n}_{C_{i_r}}^T(t+1)] = E[\mathbf{n}_{C_{i_1-1}}^T(t)\times\ldots\times\mathbf{n}_{C_{i_r-1}}^T(t)](\mathbf{P}_{i_1-1}\otimes\ldots\otimes\mathbf{P}_{i_r-1}),$$

for $i_1, i_2, \ldots, i_r = 0, 1, \ldots, d-1$ ($C_{-1} \equiv C_{d-1}$ και $\mathbf{P}_{-1} \equiv \mathbf{P}_{d-1}$).

Moments about zero

By using Stirling numbers of the second kind, for $r \in \mathbb{N}$, we have,

$$\begin{aligned} n_j^r(t+1) &= S(r,1)n_j(t+1) + S(r,2)n_j^{(2)}(t+1) + \ldots + S(r,r)n_j^{(r)}(t+1) \\ &= \sum_{i=1}^{r} S(r,i)n_j^{(i)}(t+1). \end{aligned}$$

Therefore, the rth-order moments about 0 of the random variable $n_j(t+1)$ are given by the equation:

$$E\left[n_j^r(t+1)\right] = \sum_{i=1}^{r} S(r,i)E\left[n_j^{(i)}(t+1)\right]$$

Skewness and kurtosis

It is known that,

$$\mu_r = \sum_{j=o}^{r} (-1)^j \binom{r}{j} (\mu'_1)^j \mu'_{r-j}$$

Then,

$$\begin{aligned}
\lambda_j(t+1) &= \frac{E\left[n_j(t+1) - En_j(t+1)\right]^3}{\sigma^3} \\
&= \frac{\sum_{s=0}^{3} (-1)^s \binom{3}{s} \left(En_j(t+1)\right)^s En_j^{3-s}(t+1)}{\left(En_j^{(2)}(t+1) + En_j(t+1) - (En_j(t+1))^2\right)^{3/2}}
\end{aligned}$$

and

$$\begin{aligned}
\kappa_j(t+1) &= \frac{E\left[n_j(t+1) - En_j(t+1)\right]^4}{\sigma^4} - 3 \\
&= \frac{\sum_{s=0}^{4} (-1)^s \binom{4}{s} \left(En_j(t+1)\right)^s En_j^{4-s}(t+1)}{\left(En_j^{(2)}(t+1) + En_j(t+1) - (En_j(t+1))^2\right)^{2}} - 3.
\end{aligned}$$

Distribution of the states' sizes

For a random variable X, with values x=0,1,…,N, we have that

$$P\left[X=k\right]=\frac{1}{k!}\sum_{j=0}^{N-k}\frac{(-1)^j}{j!}\mu_{(k+j)},$$

where $\mu_{(r)}$ stands for the r^{th}-order factorial moment of the random variable X.

Hence, for the closed homogeneous Markov system, the distribution of the states' sizes is given by the equation:

$$P\left[n_i(t)=n\right]=\frac{1}{n!}\sum_{j=0}^{N-n}\frac{(-1)^j}{j!}E\left[n_i^{(n+j)}(t)\right]$$

Corollary 1.2: The distribution of the states' sizes $n_i(t), i = 1, \ldots, k$ for every $t \in \mathbb{N}^+$ of a closed homogeneous Markov system of size N, is given by the equation

$$P\left[n_i(t) = n\right] = \frac{1}{n!}\sum_{j=0}^{N-n}\frac{(-1)^j}{j!}\sum_{x_1+\ldots+x_k=n+j}\frac{(n+j)!}{x_1!\ldots x_k!}E\left[\prod_{s=1}^{k} n_s^{(x_s)}(t-1)\right]\prod_{s=1}^{k} p_{si}^{x_s},$$

for $i = 1, \ldots, k$ and $n = 0, 1, \ldots, N$.

Proposition 1.3: The distribution of a discrete variable X, $X = (X_1, X_2, \ldots, X_k)$ for which $X_1 + X_2 + \cdots + X_k = N, \quad N \in \mathbb{N}^+$, satisfies the equation,

$$P(X_1 = r_1, X_2 = r_2, \ldots, X_k = r_k) = \frac{1}{\prod_{i=1}^{k} r_i!}E\left[\prod_{i=1}^{k} X_i^{(r_i)}\right].$$

Hence, for the closed homogeneous Markov system of discrete time, the distribution of the random variable $n(t) = (n_1(t), n_2(t), \ldots, n_t(t))^T$ is given by the equation,

$$P\left[n_1(t) = r_1, n_2(t) = r_2, \ldots, n_{k+1}(t) = r_k\right] = \frac{1}{\prod_{i=1}^{k} r_i!} E\left[\prod_{i=1}^{k} n_i^{(r_i)}(t)\right].$$

Corollary 1.3: The distribution of the state vector $n(t) = (n_1(t), n_2(t), \ldots, n_t(t))^T$ of a closed, homogeneous Markov system of discrete time, for every moment t=1,2,... is given by the equation

$$P\left[n_1(t) = r_1, n_2(t) = r_2, \ldots, n_{k+1}(t) = r_k\right] = \frac{1}{\prod_{i=1}^{k} r_i!} \times$$

$$\times \sum_{x_{11}+\ldots+x_{k1}=r_1} \cdots \sum_{x_{1k}+\ldots+x_{kk}=r_k} \left(\prod_{j=1}^{k} \frac{r_j!}{x_{1j}!\ldots x_{kj}!}\right) E\left[\prod_{i=1}^{k} n_i^{\left(\sum_{j=1}^{k} x_{ij}\right)}(t-1)\right] \prod_{j=1}^{k}\prod_{i=1}^{k} p_{ij}^{x_{ij}}.$$

Corollary 1.4: If the transition matrix **P** of a HMS of discrete time is fully regular, then the asymptotic distribution of the states' sizes $n_i(t), i = 1, \ldots, k$ is the Binomial with parameters N and π_i , that is

$$n_i(\infty) \sim B(N, \pi_i), \; i = 1, 2, \ldots, k,$$

where, N is the system's size and π_i are the elements of the stochastic vector $\boldsymbol{\pi}$ for which $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{P} = \boldsymbol{\pi}^{\mathrm{T}}$.

Corollary 1.5: If the transition matrix **P** of a HMS of discrete time is fully regular, then the asymptotic distribution of the states' vector $\mathbf{n}(t)$ is the Polynomial with parameters N and $\pi_1, \pi_2, \ldots., \pi_k$, that is

$$\mathbf{n}(\infty) \sim M(N, \pi_1, \pi_2, \ldots, \pi_k),$$

where, N is the system's size and π_i are the elements of the stochastic vector $\boldsymbol{\pi}$ for which $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{P} = \boldsymbol{\pi}^{\mathrm{T}}$.

Illustrative example

Let a closed discrete time HMS with 3 states and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.5 & 0.3 & 0.2 \end{pmatrix}.$$

Let N=20 and

$$\mathbf{n}(0) = (5, 3, 12)^T.$$

Then:

	$t=1$	$t=2$	$t=3$	$\ldots$	$t=\infty$
$En_1(t)$	8.6	7.07	6.872		6.7533
$En_2(t)$	6.9	8.07	8.42		8.5714
$En_3(t)$	4.5	4.86	4.707		4.6753
$var\,(n_1(t))$	4.68	4.5603	4.5097		4.4729
$var\,(n_2(t))$	4.29	4.7931	4.8735		4.898
$var\,(n_3(t))$	3.45	3.6768	3.5991		3.5824
$cov\,(n_1(1), n_2(t))$	-2.76	-2.8383	-2.892		-2.8942
$cov\,(n_1(1), n_3(t))$	-1.92	-1.722	-1.6177		-1.5787
$cov\,(n_2(1), n_3(t))$	-1.53	-1.9548	-1.9814		-2.0037
$\lambda_1(t)$	0.0522	0.1365	0.1472		0.1535
$\lambda_2(t)$	0.1445	0.0876	0.0715		0.0646
$\lambda_3(t)$	0.2903	0.2677	0.279		0.2813
$\kappa_1(t)$	-0.0917	-0.0811	-0.0783		-0.0764
$\kappa_2(t)$	-0.0676	-0.0915	-0.0948		-0.0958
$\kappa_3(t)$	-0.0149	-0.0283	-0.0222		-0.0209

Factorial moments of the states' sizes of a continuous time HMS

Proposition 1.4: Let a continuous time HMS with transition rates matrix $\boldsymbol{Q} = (q_{ij}), \mathrm{i}, \mathrm{j} = 1, \ldots, \mathrm{k}$ then the rate of the r-th-order factorial moment $E[n_j^{(r)}(t)]$ of the random variable $n_j(t)$, j=1,2,…,k is given by the relation,

$$\frac{d(E[n_j^{(r)}(t)])}{dt} = r\left(q_{jj}E[n_j^{(r)}(t)] + \sum_{\substack{i=1 \\ i\neq j}}^{k} q_{ij}E[n_j^{(r-1)}(t)n_i(t)]\right).$$

Proposition 1.5: Let a continuous time HMS with transition rates matrix $\boldsymbol{Q} = (q_{ij}), \mathrm{i}, \mathrm{j} = 1, \dots, \mathrm{k}$ then the rate of the mixed moments $E[\prod_{i=1}^{k} n_i^{(r_i)}(t)]$ of the random variable **n**(t), is given by the relation,

$$\frac{d\left(E\left[\prod_{i=1}^{k} n_i^{(r_i)}(t)\right]\right)}{dt} = E\left[\prod_{i=1}^{k} n_i^{(r_i)}(t)\right]\sum_{j=1}^{k} r_j q_{jj} +$$

$$+ \sum_{s=1}^{k}\sum_{\substack{j=1 \\ j\neq s}}^{k} r_s q_{js} E\left[n_1^{(r_1)}(t) n_2^{(r_2)}(t) \dots n_j^{(r_j+1)}(t) \dots n_s^{(r_s-1)}(t) \dots n_k^{(r_k)}(t)\right],$$

where, $r_i \in \mathbb{N}, i = 1,2, \dots, k$

Theorem 1.2: For a continuous time HMS with transition rate matrix **Q,** it holds that,

$$\frac{d}{dt}\left(E[\underbrace{\mathbf{n}^T(t)\times\ldots\times\mathbf{n}^T(t)}_{r}]\right) = E[\underbrace{\mathbf{n}^T(t)\times\ldots\times\mathbf{n}^T(t)}_{r}](\underbrace{\mathbf{I}\otimes\mathbf{I}\ldots\otimes\mathbf{Q}}_{r}$$
$$+\ \underbrace{\mathbf{I}\otimes\mathbf{I}\ldots\otimes\mathbf{Q}\otimes\mathbf{I}}_{r}+\ldots+\underbrace{\mathbf{Q}\otimes\mathbf{I}\ldots\otimes\mathbf{I}}_{r}),$$

Corollary 1.6: Let $\boldsymbol{Q} \in M_n$ and $\mathrm{r} \in \mathbb{N}$, then,

$$\exp\{\underbrace{\mathbf{I}\otimes\mathbf{I}\otimes\ldots\otimes\mathbf{Q}}_{r}+\underbrace{\mathbf{I}\otimes\mathbf{I}\otimes\ldots\otimes\mathbf{Q}\otimes\mathbf{I}}_{r}+\ldots+\underbrace{\mathbf{Q}\otimes\mathbf{I}\otimes\ldots\otimes\mathbf{I}}_{r}\} =$$
$$= \underbrace{\exp\{\mathbf{Q}\}\otimes\exp\{\mathbf{Q}\}\otimes\ldots\otimes\exp\{\mathbf{Q}\}}_{r},$$

Corollary 1.6: For a continuous time HMS, the factorial moments and the mixed factorial moments of the states' sizes at time t, are given by the relations,

$$
\begin{aligned}
E[\underbrace{\mathbf{n}^T(t) \times \ldots \times \mathbf{n}^T(t)}_{r}] &= (\underbrace{\mathbf{n}^T(0) \times \ldots \times \mathbf{n}^T(0)}_{r}) \exp\{(\underbrace{\mathbf{I} \otimes \mathbf{I} \ldots \otimes \mathbf{Q}}_{r} \\
&+ \underbrace{\mathbf{I} \otimes \mathbf{I} \ldots \otimes \mathbf{Q} \otimes \mathbf{I}}_{r} + \ldots + \underbrace{\mathbf{Q} \otimes \mathbf{I} \ldots \otimes \mathbf{I}}_{r})t\},
\end{aligned}
$$

or,

$$
E[\underbrace{\mathbf{n}^T(t) \times \ldots \times \mathbf{n}^T(t)}_{r}] = (\underbrace{\mathbf{n}^T(0) \times \ldots \times \mathbf{n}^T(0)}_{r})(\underbrace{e^{\mathbf{Q}t} \otimes \ldots \otimes e^{\mathbf{Q}t}}_{r}).
$$

Corollary 1.7: If the transition matrix **P**(t) of a closed continuous time HMS is fully regular then,

$$E[\underbrace{\mathbf{n}^T(\infty) \times \ldots \times \mathbf{n}^T(\infty)}_{r}] = N^{(r)}(\underbrace{\pi^T \otimes \pi^T \otimes \ldots \otimes \pi^T}_{r}),$$

where, $\boldsymbol{n}^T(\infty) = \lim_{t\to\infty} \boldsymbol{n}^T(t)$, N is the system's size and $\boldsymbol{\pi}$ is the stochastic vector for which $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{Q} = \mathbf{0}^T$.

Remark 1.1: From the Corollary 4.2.2, we conclude that the asymptotic distribution of the state vector of a continuous time HMS is Polynomial with parameters N and $\pi_1, \pi_2, \ldots, \pi_k$.

Remark 1.2: By means of the factorial moments, we can compute apart from the means, variances and covariances of the states' sizes, also kurtosis and skewness coefficients.

Distribution of the states' sizes

Discrete time:

$$P\left[n_i(t) = n\right] = \frac{1}{n!}\sum_{j=0}^{N-n}\frac{(-1)^j}{j!}E\left[n_i^{(n+j)}(t)\right],$$

$$P\left[n_1(t) = r_1, n_2(t) = r_2, \ldots, n_{k+1}(t) = r_k\right] = \frac{1}{\prod_{i=1}^{k} r_i!}E\left[\prod_{i=1}^{k} n_i^{(r_i)}(t)\right].$$

For the r^th order factorial moments of the states' sizes we have,

$$E[\underbrace{\mathbf{n}^T(t)\times\ldots\times\mathbf{n}^T(t)}_{r}] = (\underbrace{\mathbf{n}^T(0)\times\ldots\times\mathbf{n}^T(0)}_{r})(\underbrace{e^{\mathbf{Q}t}\otimes\ldots\otimes e^{\mathbf{Q}t}}_{r}).$$

Corollary 1.8: Let **Q** be the transition rate matrix of a continuous time HMS, then the asymptotic distribution of the states' sizes $n_i(t)$ i=1,...,k is Binomial with parameters N and π_i, that is

$$n_i(\infty) \sim B(N, \pi_i),\ i = 1, 2, \ldots, k,$$

where, N is the system's size and π_i are the elements of the stochastic vector $\boldsymbol{\pi}$ for which $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{Q} = \mathbf{0}^{\mathrm{T}}$.

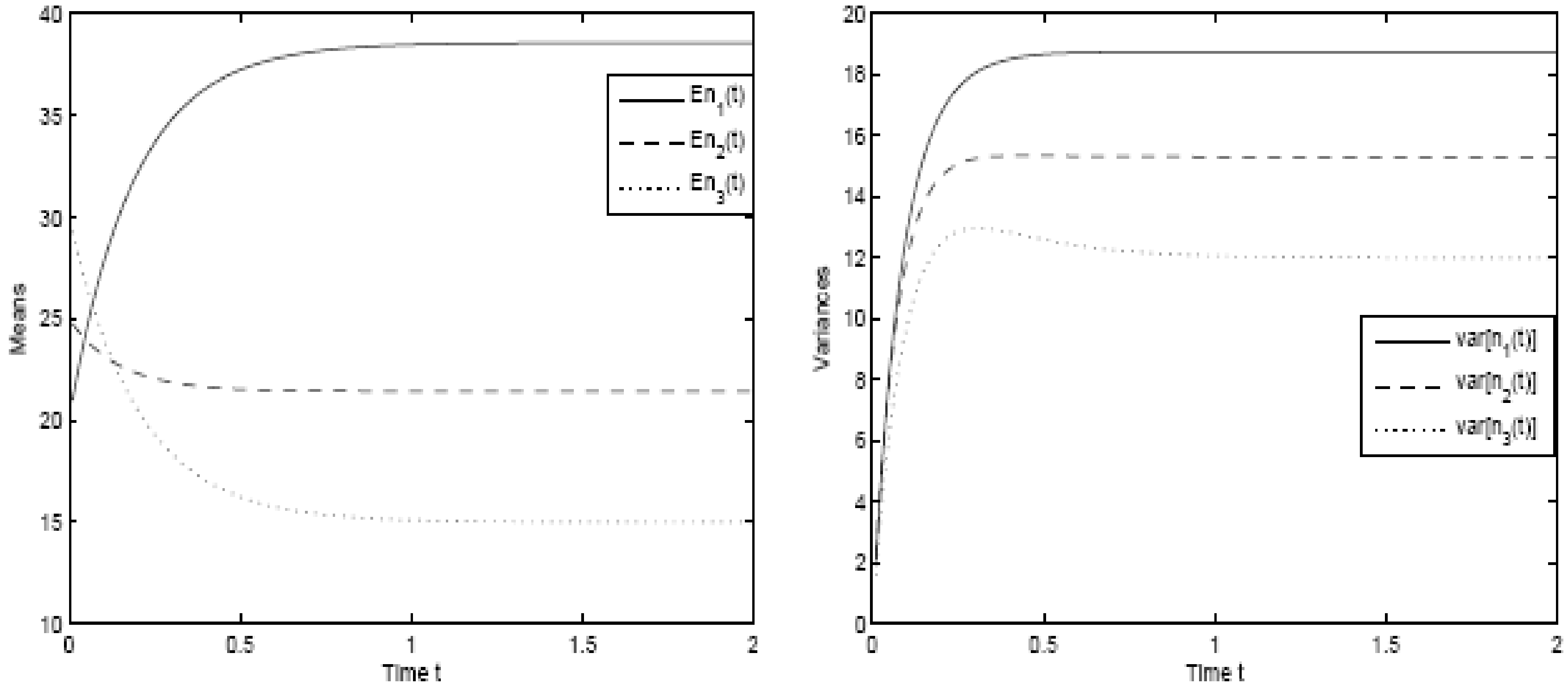
Illustrative example

Let a closed continuous time HMS with N=75 and

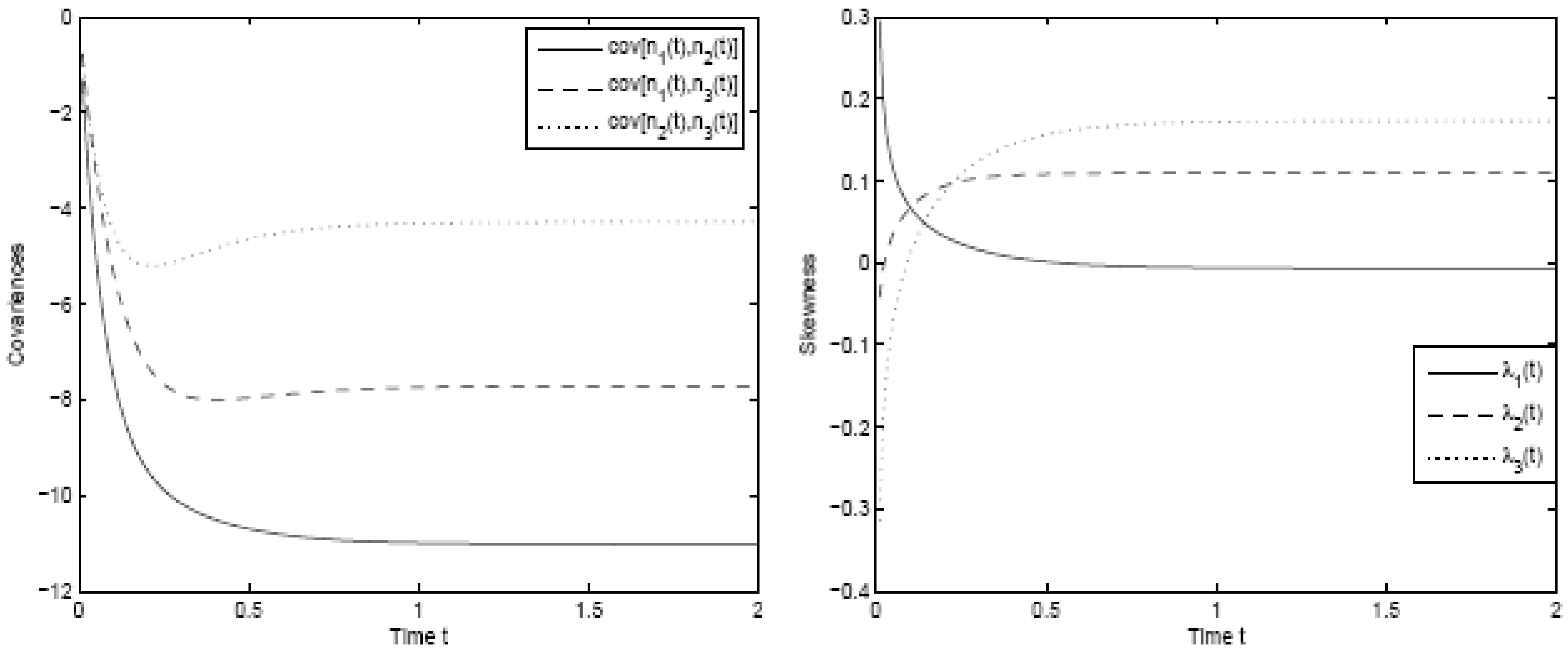
$$Q = \begin{pmatrix} -3 & 2 & 1 \\ 4 & -5 & 1 \\ 2 & 2 & -4 \end{pmatrix}.$$

and

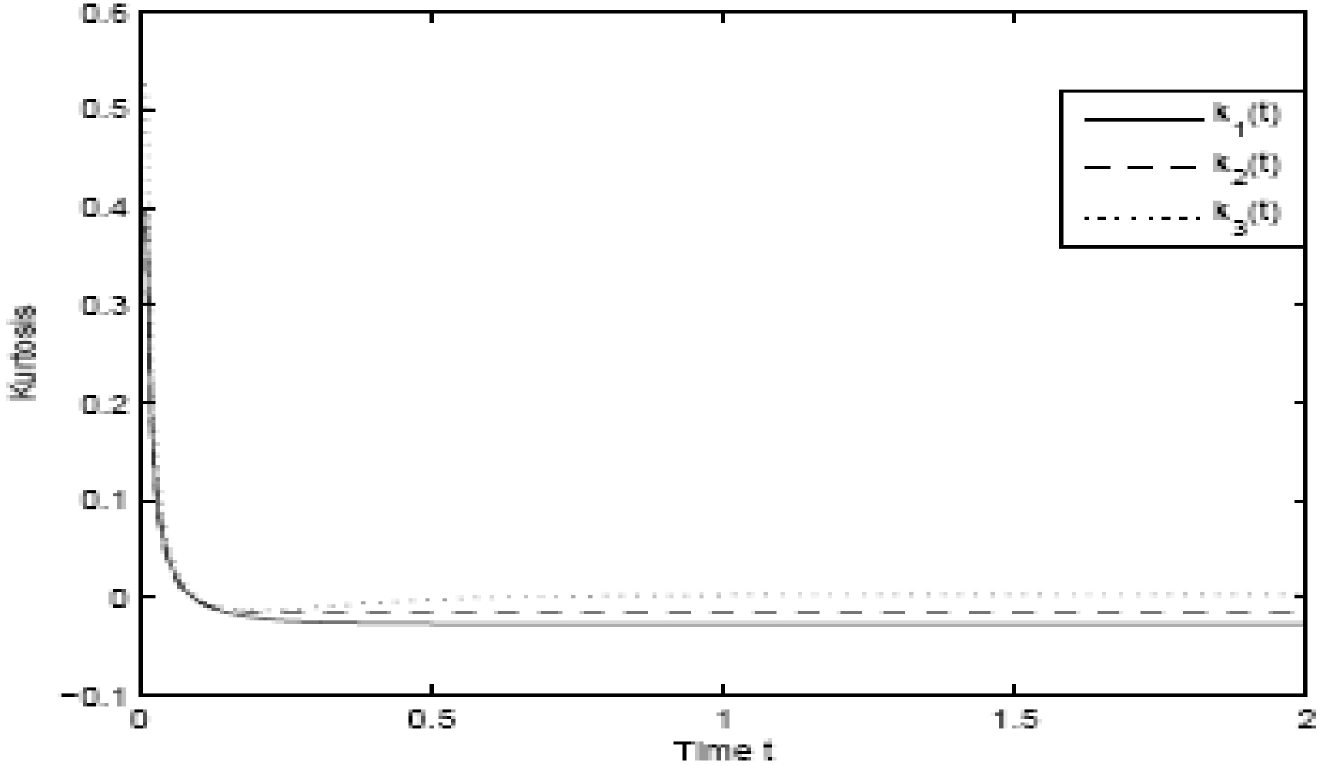
$$\mathbf{n}(0) = (20, 25, 30)^T.$$



Means
Time t
En_1(t)
En_2(t)
En_3(t)
Variances
var[n_1(t)]
var[n_2(t)]
var[n_3(t)]



0
-2
-4
-6
-8
-10
-12
Covariances
0
0.5
1
1.5
2
Time t
cov[n1(t),n2(t)]
cov[n1(t),n3(t)]
cov[n2(t),n3(t)]
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
Skewness
0
0.5
1
1.5
2
Time t
λ1(t)
λ2(t)
λ3(t)



0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
Kurtosis
0
0.5
1
1.5
2
Time t
κ1(t)
κ2(t)
κ3(t)

The continuous time closed Homogeneous Markov system (HMS) as an elastic medium. The 3-d case.

Continuous time closed Homogeneous Markov system (HMS)

Key relations

Let

$$p_{ij}(t, t+\Delta\text{t}) = q_{ij} + 0(\Delta\text{t}), \text{for i} \neq j \qquad (1.1)$$

Then

$$x_j(t+\Delta\text{t}) = \sum_{i=1}^{k} x_i(t)(\delta_{ij} + q_{ij}\Delta\text{t}) + 0(\Delta\text{t}), \qquad \text{j}\epsilon S, t \geq 0$$

From which

$$\dot{\boldsymbol{x}}'(t) = \boldsymbol{x}'(t)\boldsymbol{Q} \qquad (1.2)$$

and

$$\boldsymbol{x}'(t) = \boldsymbol{x}'(0)e^{\boldsymbol{Q}t}$$

Continuous time closed Homogeneous Markov system (HMS)

1. The consideration of the continuous medium

If we consider an HMS structure as a point of (Π), then we can consider the stochastic points of (Π) as points of a continuous medium, adopting that:

“the motion of a point at any instant moment is due to its interaction with its environment”

We note as:

$$A_t = \{\boldsymbol{x}(t)\colon\ \boldsymbol{x}'(t) = \boldsymbol{x}'(0)e^{\boldsymbol{Q}t} \quad \text{where } \boldsymbol{x}(0) \text{ is stohastic}\}$$

and if $A_n(t)$ stands for the area of (Π) that is defined by A_t, then:

$$A_n(0) \xrightarrow{\ \boldsymbol{x}'(t) = \boldsymbol{x}'(0)e^{\boldsymbol{Q}t}\ } A_n(t)$$

For n=3,

$$\mathbf{Q} = \begin{pmatrix} -q_{12} - q_{13} & q_{12} & q_{13} \\ q_{21} & -q_{21} - q_{23} & q_{23} \\ q_{31} & q_{32} & -q_{31} - q_{32} \end{pmatrix}$$

Let

$$\boldsymbol{t}^{(\boldsymbol{n})} = \lim_{\Delta S \to 0} (\Delta \mathbf{f} \,/\, \Delta S) = \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}S} \quad \text{and } \boldsymbol{t}^{(\boldsymbol{n})} = \boldsymbol{T} \cdot \boldsymbol{n}$$

Continuous time closed Homogeneous Markov system (HMS)

2. The continuous time closed Homogeneous Markov system (HMS) as an elastic medium.

The motion is taking place on the hyperplane (Π): $x_1+x_2+\ldots+x_n=1$. If matrix $\boldsymbol{Q}$ is inseparable, then eq. (1.2) has a stochastic stability point, $\boldsymbol{\pi}$. We consider, at $\boldsymbol{\pi}$, the rectangular coordinate system $\{\mathbf{f}_1, \mathbf{f}_2,\ldots,\mathbf{f}_n\}$, where $\mathbf{f}_1, \mathbf{f}_2 \ldots \mathbf{f}_{n-1}$, belong to (Π) and $\mathbf{f}_n \perp(\Pi)$.

Let,

$$\mathbf{F} = [\,\mathbf{f}_1, \mathbf{f}_2,\ldots, \mathbf{f}_n\,] = [\mathbf{F}_1 \,|\, \mathbf{f}_n\,], \text{ where } \mathbf{F}_1 = [\mathbf{f}_1 \,|\, \mathbf{f}_2 \,|\ldots|\mathbf{f}_{n-1}\,].$$

We denote by $z_1,z_2,\ldots,z_n$ the coordinates of a random point $\mathbf{z}\in(\Pi)$ with respect to the rectangular coordinate system $\{\mathbf{f}_1, \mathbf{f}_2,\ldots,\mathbf{f}_n\}$. Eq. (1.2) is considered for points of (Π) with respect to the rectangular coordinate system $\{\mathbf{f}_1, \mathbf{f}_2,\ldots,\mathbf{f}_n\}$ in the form of $\dot{\boldsymbol{z}}'(t) = \dot{\boldsymbol{z}}'(t)\cdot\boldsymbol{G}$ *or*

$$\dot{\boldsymbol{z}}'(t) = \dot{\boldsymbol{z}}'(t)\cdot\boldsymbol{G} \qquad (2.1)$$

where,

$$\boldsymbol{z} = (\mathrm{z}_1, \mathrm{z}_{\mathrm{z}2}, \ldots, \mathrm{z}_{\mathrm{n}-1})' \quad \text{and } \mathbf{G} = \mathbf{F}_1'\mathbf{Q}\mathbf{F}_1$$

<u>**Question**</u>

Can eq. (2.1) express the velocity field of a homogeneous, isotropic, linear elastic medium?

Continuous time closed Homogeneous Markov system (HMS)

2. The continuous time closed Homogeneous Markov system (HMS) as an elastic medium

Cauchy equation:

$$\rho(\mathbf{z},t)\ \mathbf{a}(\mathbf{z},t) = \mathrm{div}\mathbf{T}(\mathbf{z},t) \qquad (2.2)$$

where

- $\rho(\mathbf{z},t)$: the density at point P at time t
- $\mathbf{a}(\mathbf{z},t)$: the acceleration at P at time t with respect to the coordinate system $\{\mathbf{f}_1,\mathbf{f}_2,\ldots,\mathbf{f}_{n-1}\}$,
- $\mathrm{div}\mathbf{T} = (\sum_{j=1,2,\ldots,n-1} (\partial T_{ij}/\partial z_j))$, $i=1,2,\ldots,n-1$

<u>Density $\rho(\mathbf{z},t)$:</u>

Assuming that the HMS is isotropic, we have that $\rho(\mathbf{z},t)=\rho(t)$ and therefore it can be shown based on the continuity equation that:

$$\rho(t)=e^{-t\cdot \mathrm{tr}\mathbf{G}},\ \ t\geq 0 \qquad (2.3)$$

Because $\mathrm{tr}\mathbf{G}=\mathrm{tr}\mathbf{Q}<0$, the field (1.2) is compressible.

Remark. The rate of change of the density constitutes a measure for the rate of convergence of the (probabilistic) system to the stability point.

Continuous time closed Homogeneous Markov system (HMS)

2. The continuous time HMS of constant size as a linear elastic medium

Acceleration **a(z**,t**):**

We have, $\mathbf{a}(\mathbf{z},t)=\partial\mathbf{v}/\partial t + \nabla\mathbf{v}\cdot\mathbf{v}$

Then, $\mathbf{a}(\mathrm{t}) = \nabla\boldsymbol{v}\cdot\boldsymbol{v} = \boldsymbol{G}'\cdot\dot{\boldsymbol{z}} = (\boldsymbol{G}')^{2}\cdot\boldsymbol{z}$

Therefore,

$$\mathbf{a}(\boldsymbol{z},t) = \mathbf{a} = (\boldsymbol{G}^{2})'\cdot\boldsymbol{z}, \text{ for all t} \qquad (2.4)$$

where **z** is the position vector.

The stress tensor:

Let,

$$\mathbf{T}= \lambda\cdot \mathrm{tr}\mathbf{E}\cdot\mathbf{I} + 2\mu\mathbf{E} \qquad (2.5)$$

where,

- λ,μ: Lame constants
- $\mathbf{E}=(\varepsilon_{ij})$: (n-1)x(n-1) Euler's strain tensor

Continuous time closed Homogeneous Markov system (HMS)

2. The continuous time HMS of constant size as a linear elastic medium

The elements of the Eulerian strain tensor are

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial z_j} + \frac{\partial u_j}{\partial z_i} - \frac{\partial u_i}{\partial z_j}\frac{\partial u_j}{\partial z_i}\right) \qquad (2.6)$$

where, **u=**(u_i) is the displacement vector

Continuous time closed Homogeneous Markov system (HMS)

3. The 3-d HMS as a linear elastic medium

If n=3, then

$$\mathbf{Q} = \begin{pmatrix} -q_{12}-q_{13} & q_{12} & q_{13} \\ q_{21} & -q_{21}-q_{23} & q_{23} \\ q_{31} & q_{32} & -q_{31}-q_{32} \end{pmatrix}$$, ($q_{ij} \geq 0$).

Let

$$\mathbf{Q} = \begin{pmatrix} -4,7 & 4 & 0,7 \\ 4,02 & -4,22 & 0,2 \\ 0,2 & 2 & -2,2 \end{pmatrix}$$

and

$$\mathbf{f}_1 = \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}' \quad \mathbf{f}_2 = (0 \quad -\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}})' \quad \mathbf{f}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}'$$

Then

$$\mathbf{G} = \mathbf{F}_1'\mathbf{Q}\mathbf{F}_1 = \begin{pmatrix} -6,81 & 1.969 \\ 3,308 & -4,31 \end{pmatrix}$$

Continuous time closed Homogeneous Markov system (HMS)

3. 3-d HMS as a linear elastic medium

The eigenvalues of **Q** are λ_1=-8.41 and λ_2=-2.72. As a result, the velocity field $\dot{\mathbf{z}}' = \mathbf{z}'\mathbf{G}$ indicates a compressible medium.

Equations of motion:

$$z_1(t) = (0.72e^{-8.4t} + 0.28e^{-2.72t})z_{10} + (-0.58e^{-8.4t} + 0.58e^{-2.72t})z_{20}$$

$$z_2(t) = (-0.35e^{-8.4t} + 0.35e^{-2.72t})z_{10} + (0.28e^{-8.4t} + 0.72e^{-2.72t})$$

Shift vector: $\mathbf{u}(\mathbf{z};t,t+\Delta t) = (u_1(\mathbf{z}; t,t+\Delta t), u_2(\mathbf{z}; t,t+\Delta t))'$
where,

$$u_1(\mathbf{z};t,t+\Delta t)) = (-1 + 0.72e^{-8.4\Delta t} + 0.28e^{-2.72\Delta t})z_1 + (-0.58e^{-8.4\Delta t} + 0.58e^{-2.72\Delta t})z_2$$

$$u_2(\mathbf{z};t,t+\Delta t)) = (-0.35e^{-8.4\Delta t} + 0.35e^{-2.72\Delta t})z_1 + (0.28e^{-8.4\Delta t} + 0.72e^{-2.72\Delta t})z_2$$

From eq. (2.6) we get the strain tensor $\boldsymbol{E}=(\varepsilon_{ij})$,

$$\varepsilon_{11}(t) = -1.5 - 0.319e^{-16.8t} - 0.08e^{-11.12t} + 1.44e^{-8.4t} - 0.992e^{-5.44t} + 0.56e^{-2.72t}$$

$$\varepsilon_{12}(t) = \varepsilon_{21}(t) = 0.26e^{-16.8t} - 0.52e^{-11.12t} - 0.93e^{-8.4t} - 0.21e^{-5.44t} + 0.93e^{-2.72t}$$

$$\varepsilon_{22}(t) = -1.5 - 0.21e^{-16.8t} + 0.14e^{-11.12t} + 0.56e^{-8.4t} - 0.43e^{-5.44t} + 1.44e^{-2.72t}$$

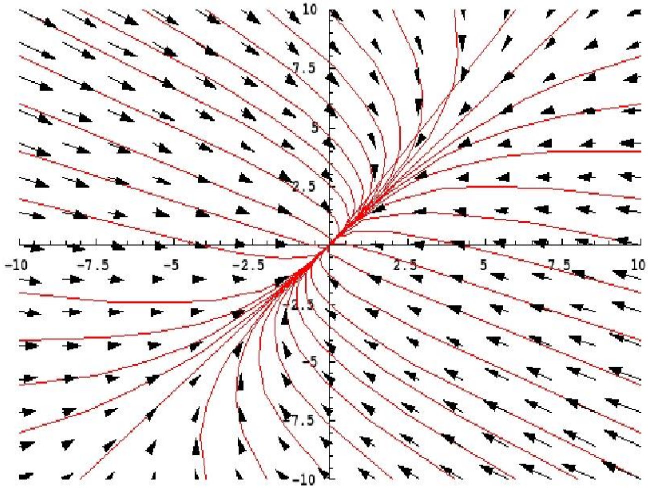
Continuous time closed Homogeneous Markov system (HMS)

3. 3-dimensional HMS as a linear elastic medium

Based on eq. (2.5), the stress tensor *T* can be calculated

The velocity field:

$$\ddot{z}_1 = 52.89z_1 - 36.79z_2$$
$$\ddot{z}_2 = -21.89z_1 + 25.09z_2$$



Continuous time closed Homogeneous Markov system (HMS)

3. 3-d HMS as a linear elastic medium

Acceleration field:

$$\ddot{z}_1 = 52.89z_1 - 36.79z_2$$
$$\ddot{z}_2 = -21.89z_1 + 25.09z_2$$

By replacing **a**(**z**,t) and **T**(**z**,t) in Cauchy's equation (2.2), we get the system of partial differential equations

$$(52.889z_1 - 36.787z_2)\rho = (-3 - 0.52e^{-16.8t} + 0.06e^{-11.12t} + 2e^{-8.4t} - 0.53e^{-8.4t} + 2e^{-2.72t})\frac{\partial\lambda}{\partial z_1} +$$
$$+2(-1.5 - 0.32e^{-16.8t} - 0.08e^{-11.12\Delta t} + 1.44e^{-8.4t} - 0.1e^{-5.44t} + 0.56e^{-2.72t})\frac{\partial\mu}{\partial z_1} +$$
$$+2(0.26e^{-16.8t} - 0.52e^{-11.12t} - 0.93e^{-8.4t} - 0.21e^{-5.44t} + 0.93e^{-2.72t})\frac{\partial\mu}{\partial z_2}$$

$$(-21.893z_1 + 25.089z_2)\rho = (-3 - 0.52e^{-16.8t} + 0.06e^{-11.12t} + 2e^{-8.4t} - 0.53e^{-8.4t} + 2e^{-2.72t})\frac{\partial\lambda}{\partial z_2}$$
$$+2(0.258e^{-16.8t} - 0.52e^{-11.12t} - 0.93e^{-8.4t} - 0.206e^{-5.44t} + 0.93e^{-2.72t})\frac{\partial\mu}{\partial z_1}$$
$$+2(-1.5 - 0.21e^{-16.8t} - 0.137e^{-11.12t} + 0.56e^{-8.4t} - 0.43e^{-5.44t} + 1.44e^{-2.72t})\frac{\partial\mu}{\partial z_2}$$

Continuous time closed Homogeneous Markov system (HMS)

3. 3-d HMS as a linear elastic medium

Based on the continuity equation of the continuous medium we have,

$$\rho(t)=e^{11.2t}, \quad t\geq 0$$

and by seeking a solution (λ,μ) of the form

$$\lambda = Z_1(z_1)T_1(t) + Z_2(z_2)T_2(t) \quad , \ \mu = K_1(z_1)T_3(t) + K_2(z_2)T_4(t)$$

we finally get

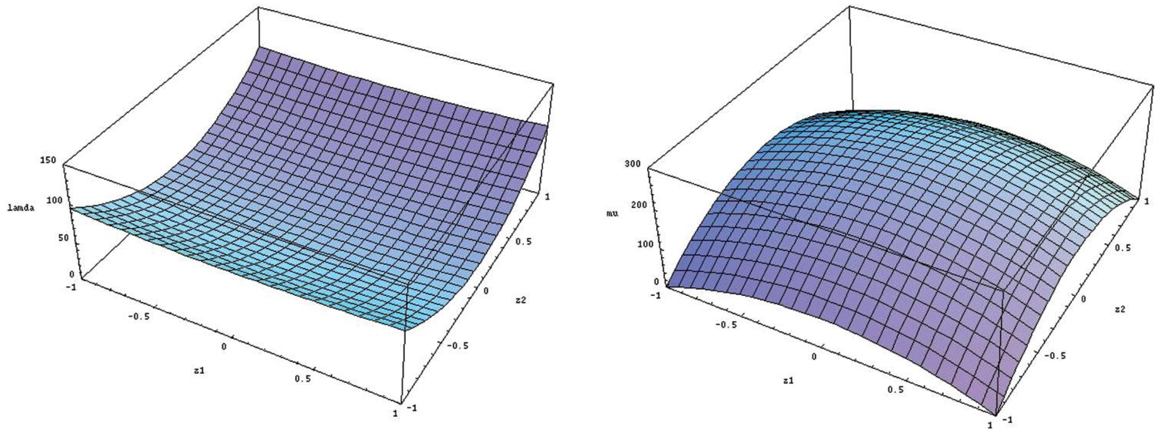
$$\begin{aligned}
&52.889 z_1 \rho(t) = (\varepsilon_{11}(t) + \varepsilon_{22}(t))\frac{dZ_1}{dz_1}T_1(t) + 2\varepsilon_{11}(t)\frac{dK_1}{dz_1}T_3(t) \\
&-36.787 z_2 \rho(t) = 2\varepsilon_{12}(t)\frac{dK_2}{dz_2}T_4(t) \\
&-21.893 z_1 \rho(t) = 2\varepsilon_{21}(t)\frac{dK_1}{dz_1}T_3(t) \\
&25.089 z_2 \rho(t) = (\varepsilon_{11}(t) + \varepsilon_{22}(t))\frac{dZ_2}{dz_2}T_2(t) + 2\varepsilon_{22}(t)\frac{dK_2}{dz_2}T_4(t)
\end{aligned} \tag{3.1}$$

Continuous time closed Homogeneous Markov system (HMS)

3. 3-d HMS as a linear elastic medium

The system of equations (3.1) are the constitutive equations of the HMS-homogeneous medium. By replacing ε_{ij} and ρ(t) we derive Lame constants as

$$\lambda(t)=\lambda(\rho(t),\boldsymbol{z},\boldsymbol{E}(t)),\qquad \mu(t)=\mu(\rho(t),\boldsymbol{z},\boldsymbol{E}(t))$$



(a) λ (b) μ

Figure 1. Lame constants for t=0.05

Continuous time closed Homogeneous Markov system (HMS)

3. 3-d HMS as a linear elastic medium

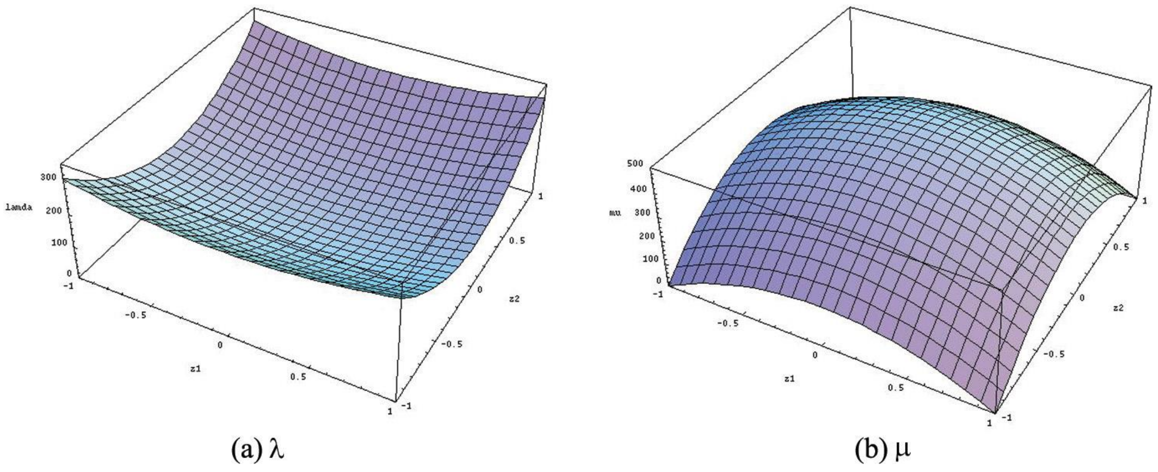


Figure 2. **Lame constants for t=0.2**

Closed time-Homogeneous Markov system (HMS)

3. 3-d HMS as a linear elastic medium

The energy of the HMS-homogeneous medium

The rate of change of the energy of the HMS-homogeneous medium is

$$\frac{dE}{dt} = \frac{dU}{dt} + \frac{dK}{dt}$$

where,

- U is the internal energy
- K is the kinetic energy

For the internal energy we have:

$$\rho(t)\frac{dU}{dt} = tr(\mathbf{GT})$$

Remark. The rates of change of the internal and the kinetic energy constitute measures of the variation of two important (energy-) components of the probabilistic system -which provide a twofold characterization of the system-, i.e. one due to the compression of the system and the other one is the translational. Apparently, they provide a two-dimensional characterization for the rate of convergence of the system.

Continuous time closed Homogeneous Markov system (HMS)

3. 3-d HMS as a linear elastic medium

Then,

$$\frac{dU}{dt} = (5.68e^{-28t} - 0.617e^{-22.32t} - 22.24e^{-19.6t} + 5.868e^{-16.63t} - 22.24e^{-13.91t} + 33.36e^{-11.2t})\lambda(t)$$
$$+ (8.875e^{-28t} - 0.617e^{-22.32t} - 34.24e^{-19.6t} + 2.87e^{-16.63t} - 10.24e^{-13.91t} + 33.36e^{-11.2t})\mu(t)$$

and

$$\frac{d\mathrm{K}}{dt} = (-5.36e^{-16.8t} - 0.9e^{-11.12t} - 0.54e^{-5.4t})z_{10}^2 + (8.67e^{-16.8t} - 1.15e^{-11.2t} - 2.24e^{-5.43t})z_{10}z_{20}$$
$$+ (-3.5e^{-16.8t} + 1.52e^{-11.2t} - 2.32e^{-5.43})z_{20}^2$$

Continuous time closed Homogeneous Markov system (HMS)

3. 3-d HMS as a linear elastic medium

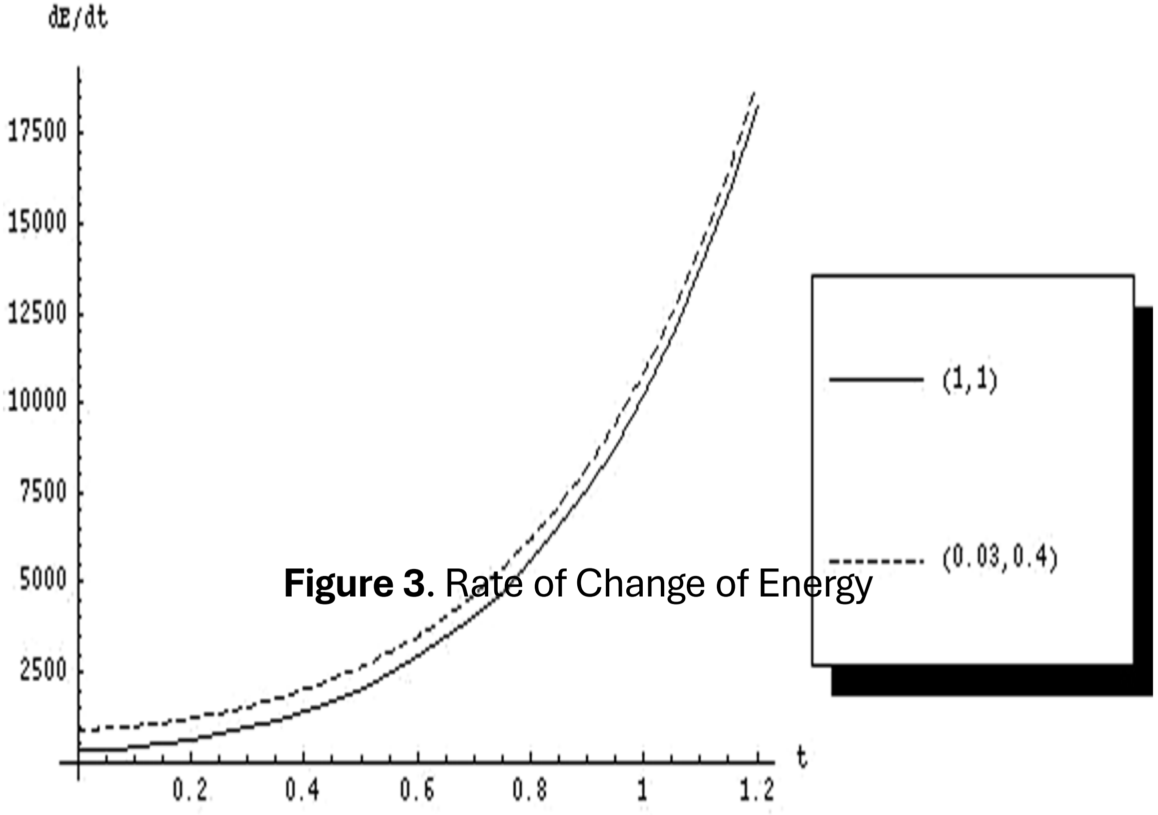


Figure 3. Rate of Change of Energy

References:

1. D. J. Bartholomew, *Stochastic models for social processes*, third edn., Wiley, New York, 1982.

2. G. S. Davies, 'Structural control in a graded manpower system', *Man.Sci*. **20**,76-84 (1973a).

3. G. S. Davies, 'Attainable and maintainable regions in Markov chain control', *Recent Theoret. Developm. in Control* , 371-381. (1978).

4. D. L. Isaacson, and R. W. Madsen, *Markov chains theory and applications*, Wiley, New York, 1976.

5. J.-O. Maaita a , G. Tsaklidis b & E. Meletlidou a 'The Homogeneous Markov System (HMS) as an Elastic Medium. The Three-Dimensional Case' *Comm. in Statistics.Theory and Methods*, 42: 2259–2270, 2013.

6.S. I. McClean, 'Continuous time stochastic models for a multigrade population', *J. Appl. Prob.* , **15**, 26-37 (1978).

7. S.I. Mclean, 'Extending the entropy stability measure for manpower planning', *J. Operat. Res. Soc.*, **37**, 1133-1138 (1986a).

8. G. Tsaklidis, 'The evolution of the attainable structures of a continuous time homogeneous Markov system with fixed size', *J. Appl. Prob.*, **33**, 34-47 (1996).

9. G. Tsaklidis, 'The continuous time homogeneous Markov system with fixed size as a Newtonian fluid (?)' Presented in the 7^{th} *Internat. Sympos. on Appl. Stochastic Models and Data Analysis*, Dublin 12-15 June, Ireland, (1995).

10. S. Vajda, *Mathematics of manpower planning*, Wiley, New York, 1978.

11. P.-C.G. Vassiliou, A. C. Georgiou and N.Tsantas, 'Control of asymptotic variability in non-homogeneous Markov systems', *J. Appl. Prob.*, **27**, 756-766 (1990).

12. P.-C.G. Vassiliou, and A. A. Papadopoulou, 'Non-homogeneous semi-Markov systems and maintainability of the state sizes', *J. Appl. Prob.*, **29**, 519-534 (1992).

13. Vasiliadis, G; Tsaklidis, G. On the distributions of the state sizes of closed continuous time homogeneous Markov systems. Methodol. Comput. Appl. Probab. 11, No 4, 561-582 (2009)
14. Vasiliadis, G; Tsaklidis, G. On the distributions of the state sizes of discrete time homogeneous Markov systems. Methodol. Comput. Appl. Probab. 10, No 1, 55-71 (2008)