On Multivariate Discrete q-Distributions

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Outline

q-Preliminaries, $0 < q < 1$

- q-Shifted factorial or q-Pochhammer symbol:

$$(\alpha;q)_n = \prod_{i=1}^{n} \left(1-\alpha q^{i-1}\right), (\alpha_1,\ldots,\alpha_m;q)_n = (\alpha_1;q)_n \cdots (\alpha_m;q)_n, (\alpha;q)_0 = 1$$

- General q-shifted factorial: $(\alpha;q)_\infty = \prod_{i=1}^{\infty} \left(1-\alpha q^{i-1}\right)$
- q-Number: $[x]_q = \frac{1-q^x}{1-q}$
- q-Factorial of x of order k: $[x]_{k,q} = [x]_q[x-1]_q \cdots [x-k+1]_q$, $k = 1, 2, \ldots$
- q-Factorial of k: $[k]_q! = [1]_q[2]_q \cdots [k]_q$, $k = 1, 2, \ldots$

- q-Binomial coefficient

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \frac{[n]_{k,q}}{[k]_q!}, \ k = 0, 1, \ldots, n$$

- Basic hypergeometric series or q-hypergeometric series

$${}_{s+1}\phi_s \left(\begin{array}{c} \alpha_1, \ldots, \alpha_{s+1} \\ b_1, \ldots, b_s \end{array} \middle| q; z \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1, \ldots, a_{s+1}; q)_k}{(b_1, \ldots, b_s; q)_k} \frac{z^k}{(q;q)_k}$$

- q-Binomial formula

$$\prod_{i=1}^{n} (1 + tq^{i-1}) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q t^k = {}_1\phi_0 \left(\begin{array}{c} q^{-n} \\ - \end{array} \middle| q; -q^n t \right)$$

- Small q-exponential function

$$e_q(t) = \prod_{i=1}^{\infty} (1 - t(1-q)q^{i-1}t)^{-1} = \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!} = {}_1\phi_0 \left(\begin{array}{c} 0 \\ - \end{array} \middle| q; (1-q)t \right)$$

q-Binomial Distribution of the 1^{st} kind

A. Kemp and C. Kemp (1991) defined a q-analogue of the binomial distribution with probability function in the form

$$f_X(x) = \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^{n} (1 + \theta q^{j-1})^{-1}, \ x = 0, 1, \ldots, n,$$

where $\theta > 0, \ 0 < q < 1$.

Heine distribution

A. Kemp and C. Newton (1990) showed that the limit of the pf. of the q-Binomial distribution of the 1^{st} kind, as $n \to \infty$, is the pf. of the *Heine distribution*

$$\lim_{n\to\infty} \binom{n}{x}_q q^{\binom{x}{2}} \theta^x \prod_{j=1}^{n} (1 + \theta q^{j-1})^{-1} = e_q(-\lambda) \frac{q^{\binom{x}{2}} \lambda^x}{[x]_q!}, \ x = 0, 1, \ldots,$$

for $0 < \lambda < \infty$ and $0 < q < 1$, with $\lambda = \theta/(1-q)$

Basic Hypergeometric Series

A. Kemp introduced and studied various forms of discrete q-distributions associated with basic hypegeometric series

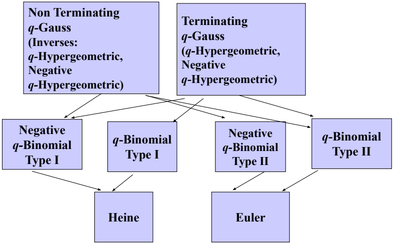
Interpetation

A. Kemp derived discrete q-distributions as stationary distributions of birth and death processes.

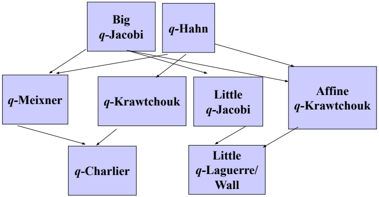
Univariate Discrete q-Distributions (Charalambides, 2016)

Univariate discrete q-distributions are based on stochastic models of sequences of n independent Bernoulli trials with success probability varying geometrically, with rate q, either with the number of previous trials or with the number of previous successes or both with the number of previous trials and successes.

Association with basic orthogonal polynomials (A.Kyriakoussis & M.V., 2010, 2012)



(a) Discrete q-Distributions

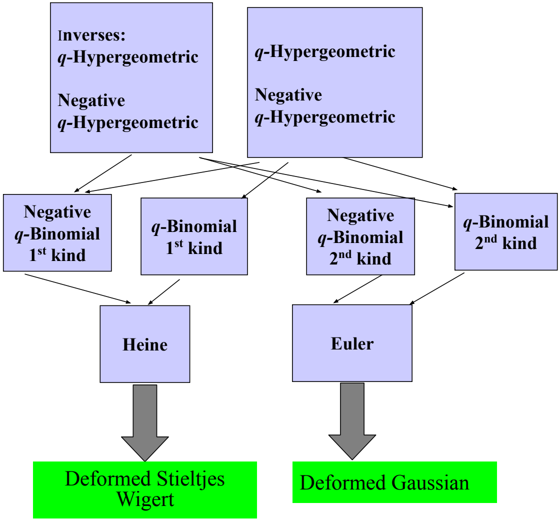


(b) Basic Orthogonal Polynomials

Asymptotic Normality and Classical Asymptotic Methods Do not Hold for Discrete q-Distributions

- Discrete q-distributions have finite mean and variance when $n \to \infty$
- Asymptotic normality does not hold
- Asymptotic methods—classical central or/and local limit theorems—as in Bender (1973), Canfield (1977), Flajolet and Soria (1990), Odlyzko (1995) et al. can not be applied
- What is the asymptotic behaviour for $n \to \infty$ of the discrete q-distributions?

Asymptotic Behaviour (Kyriakoussis & M.V., 2013, 2017, 2019, M.V. 2022, 2024)



Stieltjes-Wigert Distribution

The continuous Stieltjes–Wigert distribution has probability density function

$$v_W^{SW}(w) = \frac{q^{1/8}}{\sqrt{2\pi \log q^{-1}\, w}} e^{\frac{(\log w)^2}{2 \log q}}, \quad w > 0,$$

with mean value $\mu^{SW} = q^{-1}$ and standard deviation $\sigma^{SW} = q^{-3/2}(1-q)^{1/2}$.

Stieltjes-Wigert Distribution as Limiting Distribution of Univariate Discrete q-Distributions (K&M.V., 2013, 2019)

Transformation

From the r.v. X of the q-Binomial distribution of the 1st kind to the equal-distributed deformed r.v. $Y = [X]_{1/q} = \left(1 - q^{-X}\right) / \left(1 - q^{-1}\right)$.

q-Mean, q-Variance

$$
\begin{aligned}
\mu_q &= E\left([X_1]_{1/q}\right) = [n]_q \frac{\theta}{1+\theta q^{n-1}}, \\
(\sigma_q)^2 &= V\left([X_1]_{1/q}\right) \\
&= \frac{1-q}{q}[n]_q^2 \frac{\theta^2}{(1+\theta q^{n-1})^2(1+\theta_1 q^{n-2})} \\
&\quad + [n]_q \frac{\theta}{(1+\theta q^{n-1})(1+\theta q^{n-2})}
\end{aligned}
$$

A Stieltjes-Wigert Distribution as Limiting Distribution of q-Binomial of the 1st Kind

Theorem

Let $\theta = \theta_n$, for $n = 0, 1, 2, \ldots$, such that $\theta_n = q^{-\alpha n}$, a constant and $0 < q < 1$. Then, for $n \to \infty$, the q-Binomial distribution of the first kind is approximated by a deformed standardized Stieltjes–Wigert distribution as follows:

$$f_X^B(x) \cong \frac{q^{-7/8}}{\sigma_q (2\pi)^{1/2}} \left(\frac{\log q^{-1}}{q^{-1} - 1} \right)^{1/2} \left(q^{-3/2} (1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right)^{-1/2} q^{-x}$$
$$\cdot \exp\left(\frac{1}{2 \log q} \log^2 \left(q^{-3/2} (1-q)^{1/2} \frac{[x]_{1/q} - \mu_q}{\sigma_q} + q^{-1} \right) \right), \quad x \geq 0,$$

Stieltjes-Wigert Distribution as Limiting Distribution of Heine

Remark

A similar asymptotic result holds for the p.f. of the Heine distribution when $\lambda \to \infty$.

Univariate Absorption Distribution (A. Kemp(1998), Charalambides (2012, 2016))

Consider a sequence of independent geometric sequences of trials with probability of success at the jth geometric sequence of trials given by

$$p_j = 1 - q^{r-j+1}, \;\; j = 1, 2, \ldots, [r], \;\; 0 < r < \infty, \;\; 0 < q < 1,$$

which is a geometrically decreasing sequence of a finite number of terms. Then the probability function of the number Y_n of successes in n independent Bernoulli trials is given by

$$f_{Y_n}(y) = P(Y_n = y) = \binom{n}{y}_q q^{(n-y)(r-y)}(1-q)^r [r]_{y,q}, \;\; y = 0, 1, \ldots, n,$$

for $0 < r < \infty$, $0 < q < 1$, and $n \leq [r]$. This discrete q-distribution is known as *absorption distribution*.

q-Mean, q-Variance

$$\mu_q^A = E\left([Y]_q\right) = (1-q)[n]_q[r]_q,$$

$$\begin{aligned}(\sigma_q^A)^2 &= V\left([Y]_q\right) \\ &= (1-q)^2[n]_{2,q}[r]_{2,q} - (1-q)^2[n]_q^2[r]_q^2 + (1-q)[n]_q[r]_q.\end{aligned}$$

Asymptotic Behaviour of Univariate Absorption Distribution (M.V., 2024)

Theorem

Let $q = q(n)$ *with* $q(n) \to 1$*, as* $n \to \infty$*,* $q(n)^n = \Omega(1)$ *and* $r = O(n)$*. Then, for* $n \to \infty$*, the univariate absorption distribution is approximated by a deformed standardized Gaussian distribution as follows:*

$$f_Y(y) \cong \frac{(\log q^{-1})^{1/2}}{\sigma_q^A (2\pi(1-q))^{1/2}} \, q^y \exp\left(-\frac{1-q}{2\log q^{-1}} \left(\frac{[y]_q - \mu_q^A}{\sigma_q^A} \right)^2 \right), \quad y \geq 0.$$

Sketch Proof

- $Z = \frac{[Y]_q - \mu_q^A}{\sigma_q^A}$
- q-Stirling type $0 < q < 1$ (Kyriakoussis and M.V, 2013)

$$[n]_q! = \frac{q^{-1/8}(2\pi(1-q))^{1/2}}{(q \log q^{-1})^{1/2}} \frac{q^{\binom{n}{2}} q^{-n/2} [n]_{1/q}^{n+1/2}}{\prod_{j=1}^{\infty}(1+(q^{-n}-1)q^{j-1})} \left(1 + O(n^{-1})\right),$$

$$[n]_q! = [1]_q[2]_q \cdots [n-1]_q[n]_q \text{ with } [n]_q = \frac{1-q^n}{1-q}, \ n \geq 1.$$

- Pointwise convergence techniques applied to the probability function

Remark: Possible realizations of the sequence $q := q(n)$

$$q(n) = 1 - \frac{\alpha}{n}, \quad \alpha > 0 \quad \text{or} \quad q(n) = 1 - 1/\exp n.$$

Univariate q-Hypergeometric Distribution (A. Kemp (2005), Charalambides (2012, 2016), Kyriakoussis & M.V.(2012))

Consider an urn containing r white balls and s black balls. Let W_n be the number of white balls drawn in n q-drawings in a q-hypergeometric urn model, with the conditional probability of drawing a white ball at the q-drawing, given that $j-1$ white balls are drawn in the previous $i-1$ q-drawings given by

$$p_{i,j} = \frac{[r-j+1]_q}{[r+s-i+1]_q}, j = 1, 2, \ldots, \min\{i, r\}, i = 1, 2, \ldots, r+s.$$

The distribution of the random variable W_n is called *q-hypergeometric distribution*, with parameters n, r, s and q and its p.f. is given by

$$f_{W_n}(w_n) = P(W_n = w) = \binom{n}{w}_q q^{(n-w)(r-w)} \frac{[r]_{w,q}[s]_{n-w,q}}{[r+s]_{n,q}},$$

for $w = 0, 1, 2, \ldots, n$, where $0 < q < 1$, and r and s are positive integers.

q-Mean, q-Variance

$$\mu_q^H = E\left([W_n]_q\right) = \frac{[n]_q[r]_q}{[r+s]_q},$$

$$(\sigma_q^H)^2 = V\left([W_n]_q\right)$$

$$= q\frac{[n]_{2,q}[r]_{2,\,q}}{[r+s]_{2,q}} + \frac{[n]_q[r]_q}{[r+s]_q} - \left(\frac{[n]_q[r]_q}{[r+s]_q}\right)^2.$$

Asymptotic Behaviour of Univariate q-Hypergeometric Distribution

Theorem

Let $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $r + s = O(n)$. Then, for $n \to \infty$, the univariate q-Hypergeometric distribution is approximated by a deformed standardized Gaussian distribution as follows:

$$f_W(w) \cong \frac{(\log q^{-1})^{1/2}}{\sigma_q^H (2\pi(1-q))^{1/2}} \, q^w \exp\left(-\frac{1-q}{2\log q^{-1}} \left(\frac{[w]_q - \mu_q^H}{\sigma_q^H} \right)^2 \right), \; w \geq 0.$$

Multivariate Discrete q-Distributions (Charalambides, 2021, 2022, 2023)

Multivariate discrete q-distributions are based on stochastic models of sequences of n independent Bernoulli trials with chain-composite successes, where the odds of success of a certain kind at a trial is assumed to vary geometrically, with rate q, with the number of previous trials or with the number of previous successes or both with the number of previous trials and successes.

q-Multinomial Distribution of the 1st Kind (Charalambides, 2022)

- X_j : number of successes of a jth kind in a sequence of n independent Bernoulli trials with chain composite failures, where the probability of success of the jth kind at the ith trial is given by

$$p_{j,i} = \frac{\theta_j q^{i-1}}{1+\theta_j q^{i-1}},\ 0 < \theta_j < \infty,\ j = 1, 2, \ldots, i = 1, 2, \ldots, 0 < q < 1.$$

- Joint probability function of the random vector $\mathcal{X} = (X_1, X_2, \ldots, X_k)$ is given by

$$\begin{aligned} f_{\mathcal{X}}^{B}(x_1, x_2, \ldots, x_k) &= P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) \\ &= \binom{n}{x_1, x_2, \ldots, x_k}_q \prod_{j=1}^{k} \frac{\theta_j^{x_j} q^{\binom{x_j}{2}}}{\prod_{i=1}^{n-s_{j-1}}(1+\theta_j q^{i-1})} \end{aligned}$$

$x_j = 0, 1, 2, \ldots, n$, with $\sum_{j=1}^{k} x_j \leq n$, $s_j = \sum_{i=1}^{j} x_i$, $j = 1, 2, \ldots, k$.

Multiple Heine Distribution (Charalambides, 2021)

The discrete limit of the joint p.f. of the q-multinomial distribution of the 1st kind, as $n \to \infty$, is the joint p.f. of the *multiple Heine distribution*,

$$\lim_{n\to\infty} \binom{n}{x_1, x_2, \ldots, x_k}_q \prod_{j=1}^{k} \frac{\theta_j^{x_j} q^{\binom{x_j}{2}}}{\prod_{i=1}^{n-s_{j-1}} (1+\theta_j q^{i-1})} =$$

$$\prod_{j=1}^{k} \frac{q^{\binom{x_j}{2}} \lambda_j^{x_j}}{[x_j]_q!} \prod_{i=1}^{\infty} (1+\lambda_j(1-q)q^{i-1})^{-1},$$

$x_j = 0, 1, 2, \ldots,\ \lambda_j > 0,\ 0 < q < 1,\ \lambda_j = \theta_j/(1-q),\ j = 1, 2, \ldots, k$

Multiple Heine Distribution

Let $\mathcal{X} = (X_1, X_2, \ldots, X_k)$ be a random vector that follows the multiple Heine distribution. Then the joint p.f. the multiple Heine distribution is given by

$$f_{\mathcal{X}}^{H}(x_1, x_2, \ldots, x_k) = \prod_{j=1}^{k} \frac{q^{\binom{x_j}{2}} {\lambda_j}^{x_j}}{[x_j]_q!} \prod_{i=1}^{\infty} (1 + \lambda_j (1-q) q^{i-1})^{-1},$$

where $x_j = 0, 1, 2, \ldots,\ \lambda_j > 0,\ 0 < q < 1,\ \lambda_j = \theta_j/(1-q),\ j = 1, 2, \ldots, k,\ k \geq 2.$

q-Multinomial distribution of the 2nd kind (Charalambides, 2021)

- Y_j: number of successes of a jth kind in a sequence of n independent Bernoulli trials with chain composite failures, where the conditional probability of success of the jth kind at any trial, given that $i-1$ successes of the jth kind occur in the previous trials is given by

$$p_{j,i} = 1 - \theta_j q^{i-1}, j = 1, 2, \ldots, i = 1, 2, \ldots, 0 < \theta_j < 1, 0 < q < 1.$$

- Joint probability function of the random vector $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k)$:

$$f_{\mathcal{Y}}^{MS}(y_1, y_2, \ldots, y_k) = P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k)$$
$$= \binom{n}{y_1, y_2, \ldots, y_k}_q \prod_{j=1}^{k} \theta_j^{y_j} \prod_{i=1}^{n-s_j} \left(1 - \theta_j q^{i-1}\right),$$

$y_j = 0, 1, \ldots, n$, $\sum_{j=1}^{k} x_j \leq n$, $s_j = \sum_{i=1}^{j} y_i$, $0 < q < 1$.

Multiple Euler Distribution (Charalambides, 2021)

The discrete limit of the joint p.f. of the q-multinomial distribution of the 2nd kind, as $n \to \infty$, is the joint p.f. of the *multiple Euler distribution*

$$\lim_{n\to\infty} \binom{n}{y_1, y_2, \ldots, y_k}_q \prod_{j=1}^{k} \theta_j^{y_j} \prod_{i=1}^{n-s_j} \left(1 - \theta_j q^{i-1}\right)$$

$$= \prod_{j=1}^{\infty} \left(1 - \lambda_j(1-q)q^{j-1}\right) \frac{\lambda_j^{y_j}}{[y_j]_q!},$$

$y_j = 0, 1, \ldots$, $0 < \lambda_j < 1/(1-q)$, $0 < q < 1$, $\lambda_j = \theta_j/(1-q)$, $j = 1, 2, \ldots$.

Multivariate Absorption Distribution (M.V, 2020, Charalambides, 2022)

- Success probability at the jth kind for $1 < q < \infty$, $q \to q^{-1}$, $0 < q < 1$, $\theta_j = q^{m_j}$:

$$p_{j,i} = 1 - q^{m_j - i + 1}, \quad , 0 < m_j < \infty, \; j = 1, 2, \ldots, k, \quad i = 1, 2, \ldots, [m_j],$$

- Joint probability function of the random vector $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_k)$:

$$\begin{aligned} f_{\mathcal{Y}}(y_1, y_2, \ldots, y_k) &= P(Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k) \\ &= \binom{n}{y_1, y_2, \ldots, y_k}_q q^{\sum_{j=1}^{k}(n-s_j)(m_j-y_j)} \prod_{j=1}^{k} (1-q)^{y_j} [m_j]_{y_j,q}, \end{aligned}$$

$y_j = 0, 1, 2, \ldots, n$, $\sum_{j=1}^{k} y_j \leq n$, $s_j = \sum_{i=1}^{j} y_i$,
$0 < m_j < \infty$, $0 < q < 1$, $n \leq [m_j]$, $j = 1, 2, \ldots, k$.

Multivariate q-Hypergeometric Distribution (M.V., 2020, Charalambides, 2022)

Consider an urn containing ν balls, $\{b_1, b_2, \ldots, b_\nu\}$, of $k+1$ different ordered colors, with ν_l distinct balls of color c_l, $\{b_{s_{l-1}+1}, b_{s_{l-1}+2}, \ldots, b_{s_l}\}$, for $l = 1, 2, \ldots, k+1$, where $s_0 = 0$, and

$$s_l = \sum_{i=1}^{l} \nu_i, \quad \text{for } l = 1, 2, \ldots, k+1,$$

with $s_{k+1} = \nu$. The first k colors, $\{c_1, c_2, \ldots, c_k\}$, may be considered as shades of white, and color $\{c_{k+1}\}$ as black.

Multivariate q-Hypergeometric Distribution

W_l : number of balls of color c_l drawn in n q-drawings in a multiple q-hypergeometric urn model, with the conditional probability of drawing a ball of color c_l at the ith q-drawing, given that $j_l - 1$ white balls of color c_l and a total of colors $c_1, c_2, \ldots, c_{l-1}$ are drawn in the previous $i-1$ q-drawings:

$$p_{i,j_l}(l) = \frac{[\nu_l - j_l + 1]_q}{[\nu + s_l - i + 1]_q}, j_l = 1, 2, \ldots, i, l = 1, 2, \ldots, k, i = 1, 2, \ldots.$$

Multivariate q-Hypergeometric Distribution

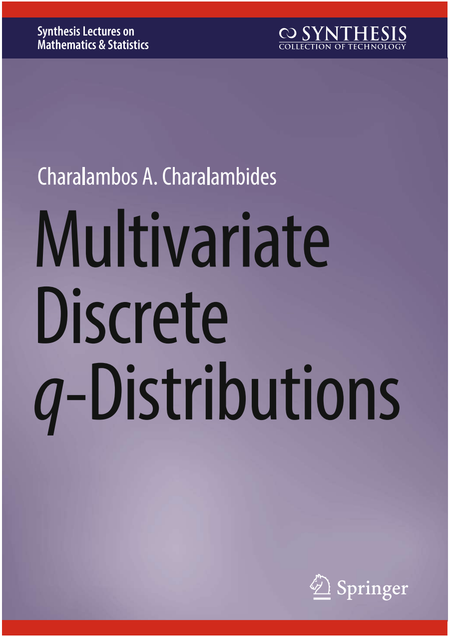
Joint probability function of a random vector $\mathcal{W} = (W_1, W_2, \ldots, W_k)$:

$$f_{\mathcal{W}}(w_1, w_2, \ldots, w_k) = P(W_1 = w_1, W_2 = w_2, \ldots, W_k = w_k)$$
$$= \binom{n}{w_1, w_2, \ldots, w_k}_q q^{\sum_{j=1}^{k}(n-\sum_{i=1}^{j} w_i)(\nu_j - w_j)} \frac{\prod_{j=1}^{k+1}[\nu_j]_{w_j,q}}{[\nu]_{n,q}}$$

$w_j = 0, 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, k$, with $\sum_{j=1}^{k} n_j \leq n$, where
$w_k = n - \sum_{j=1}^{k} w_j$,$\nu = \sum_{j=1}^{k+1} \nu_j$, $0 < q < 1$, and $n \leq [\nu_j]$, $j = 1, 2, \ldots, k$.

Remark (M.V., 2020)

- The multivariate absorption distribution emerges as a conditional distribution of a Heine process at a finite sequence of q-points in a time interval.
- The multivariate q-hypergeometric distribution arises as a conditional distribution of the multivariate absorption.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Asymptotic Behaviour of q-Multinomial Distribution of the 1st kind

Marginal probability function of X_1 : q-Binomial of the 1st kind

Marginal probability functions of X_i, $i = 2, \ldots, k$, $k \geq 2$: Not q-Binomial of the 1st kind

q-means and q-variances of random variables X_i, $i = 2, \ldots, k$, $k \geq 2$ cannot be found explicitly.

Distributions of the conditional r.vs $X_2|X_1, X_3|(X_1, X_2), \ldots, X_k|(X_1, \ldots, X_{k-1})$: q-Binomials

Conditional q-means, q-variances:

$$\mu_{[X_2]_q|X_1}, \mu_{[X_3]_q|(X_1,X_2)}, \cdots, \mu_{[X_k]_q|(X_1,\ldots,X_{k-1})},$$

$$\sigma^2_{[X_2]_q|X_1}, \sigma^2_{[X_3]_q|(X_1,X_2)}, \cdots, \sigma^2_{[X_k]_q|(X_1,\ldots,X_{k-1})}$$

Asymptotic Behaviour of q-Multinomial Distribution of the 1st kind

Conditional means and variances of the deformed r.vs $[X_j]_{1/q}$ given $X_1 = x_1, \ldots, X_{j-1} = x_{j-1}$, $j = 2, \ldots, k$, $k \geq 2$:

$$\begin{aligned}
\mu_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})} &= E\left([X_j]_{1/q}|(X_1,\ldots,X_{j-1})\right) \\
&= [n-s_{j-1}]_q \frac{\theta_j}{1+\theta_j q^{n-s_{j-1}-1}}, \\
\sigma^2_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})} &= V\left([[X_j]_{1/q}|(X_1,\ldots,X_{j-1})\right) \\
&= \frac{1-q}{q}[n-s_{j-1}]_q^2 \frac{\theta_j^2}{(1+\theta_j q^{n-s_{j-1}-1})^2(1+\theta_j q^{n-s_{j-1}-2})} \\
&\quad +[n-s_{j-1}]_q \frac{\theta_j}{(1+\theta_j q^{n-s_{j-1}-1})(1+\theta_j q^{n-s_{j-1}-2})},
\end{aligned}$$

Note

Conditional q-means, $\mu_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})}$, $3 \leq j \leq k$, $k \geq 3$: *q-regression hyperplanes*.

Asymptotic Behaviour of q-Multinomial Distribution of the 1st kind

A Multivariate Stieltjes-Wigert Distribution as a Limiting Distribution

Theorem

Let $\theta_j = \theta_{j,n} = q^{-\alpha_j n}$ with $0 < a_j < 1$, $j = 1, 2, \ldots, k$ constants and $0 < q < 1$. Then, for $n \to \infty$, the q-multinomial distribution is approximated by a deformed multivariate standardized continuous Stieltjes-Wigert distribution distribution as follows:

$$f_{\mathcal{X}}^{B}(x_1,x_2,\ldots,x_k) \cong \left(\frac{q^{-7/8}(\log q^{-1})^{1/2}}{(2\pi)^{1/2}(q^{-1}-1)^{1/2}}\right)^{k} \frac{q^{-\sum_{j=1}^{k} x_j}}{\sigma_{[X_1]_{1/q}} \prod_{j=2}^{k} \sigma_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})}}$$
$$\cdot \left(q^{-3/2}(1-q)^{1/2}\frac{[x_1]_{1/q}-\mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_{1/q}}}+q^{-1}\right)^{-1/2}$$
$$\cdot \prod_{j=2}^{k}\left(q^{-3/2}(1-q)^{1/2}\frac{[x_j]_{1/q}-\mu_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})}}{\sigma_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})}}+q^{-1}\right)^{-1/2}$$
$$\cdot \exp\left(\frac{1}{2\log q}\left(\log^2\left(\frac{(1-q)^{1/2}}{q^{3/2}}\frac{[x_1]_{1/q}-\mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_{1/q}}}+q^{-1}\right)\right)\right)$$
$$\cdot \exp\left(\frac{1}{2\log q}\sum_{j=2}^{k}\log^2\left(\frac{(1-q)^{1/2}}{q^{3/2}}\frac{[x_j]_{1/q}-\mu_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})}}{\sigma_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})}}+q^{-1}\right)\right),$$
$$x_j \geq 0, j=1,2,\ldots,k, k\geq 2. \qquad (1)$$

Sketch Proof

- $Z_1 = \frac{[X_1]_{1/q} - \mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_q}}$
- $Z_j = \frac{[X_j]_{1/q} - \mu_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})}}{\sigma_{[X_j]_{1/q}|(X_1,\ldots,X_{j-1})}}, j = 2, \ldots, k, \ \ k \geq 3$
- q-Stirling type $0 < q < 1$
- Pointwise convergence techniques applied to the joint probability function

Remark: Asymptotic Behaviour of Multiple Heine Distribution

For $\lambda_j \to \infty, \ j = 1, 2, \ldots, k$, the multiple Heine distribution is approximated by a deformed multivariate standardized continuous Stieltjes-Wigert distribution distribution as follows:

$$
\begin{aligned}
f_{\mathcal{X}}^{H}(x_1, x_2, \ldots, x_k) \ \cong \ & \left(\frac{q^{-7/8} (\log q^{-1})^{1/2}}{(2\pi)^{1/2} (q^{-1} - 1)^{1/2}} \right)^k \frac{q^{-\sum_{j=1}^{k} x_j}}{\prod_{j=1}^{k} \sigma_{[X_j]_{1/q}}} \\
& \cdot \prod_{j=1}^{k} \left(q^{-3/2} (1-q)^{1/2} \frac{[x_j]_{1/q} - \mu_{[X_j]_{1/q}}}{\sigma_{[X_j]_{1/q}}} + q^{-1} \right)^{-1/2} \\
& \cdot \exp \left(\frac{1}{2 \log q} \left(\sum_{j=1}^{k} \log^2 \frac{(1-q)^{1/2}}{q^{3/2}} \frac{[x_j]_{1/q} - \mu_{[X_j]_{1/q}}}{\sigma_{[X_j]_{1/q}}} \right) \right), \\
& x_j \geq 0, j = 1, \ldots, k, k \geq 2,
\end{aligned}
$$

$\mu_{[X_j]_{1/q}} = E\left([X_j]_{1/q}\right) = \lambda_j$ and $\sigma^2_{[X_j]_{1/q}} = V\left[X_j]_{1/q}\right) = \lambda_j q^{-1}(1-q) + \lambda_j$, $j = 1, 2, \ldots, k, k \geq 2$.

Asymptotic Behaviour of q-Trinomial Distribution

Let (X_1, X_2) be the discrete bivariate random variable that follows the q-trinomial distribution. Then the joint probability function is given by

$$
\begin{aligned}
f^{B}_{X_1,X_2}(x_1,x_2) &= P\left(X_1,X_2\right) \\
&= \binom{n}{x_1,x_2}_q \frac{\theta_1^{x_1}\theta_2^{x_2}q^{\binom{x_1}{2}+\binom{x_2}{2}}}{\prod_{i=1}^{n}(1+\theta_1 q^{i-1})\prod_{i=1}^{n-x_1}(1+\theta_2 q^{i-1})}
\end{aligned}
$$

$x_j = 0, 1, 2, \ldots, n$, $j = 1, 2$, $x_1 + x_2 \leq n$, $\theta_1, \theta_2 > 0$ and $0 < q < 1$.

Marginal probability function of X_1 : q-Binomial of the 1st kind

q-Mean and q-variance:

$$\mu_{[X_1]_{1/q}} = E\left([X_1]_{1/q}\right) = [n]_q \frac{\theta_1}{1+\theta_1 q^{n-1}}$$

and

$$\begin{aligned}(\sigma_{[X_1]_{1/q}})^2 &= V\left([X_1]_{1/q}\right) \\ &= \frac{1-q}{q}[n]_q^2 \frac{\theta_1^2}{(1+\theta_1 q^{n-1})^2(1+\theta_1 q^{n-2})} \\ &\quad + [n]_q \frac{\theta_1}{(1+\theta_1 q^{n-1})(1+\theta_1 q^{n-2})}\end{aligned}$$

Marginal probability function of X_2: Not q-Binomial of the 1st kind

Distribution of the conditional r.v. $X_2|X_1$: q-Binomial

Conditional q-mean, q-variance:

$$\begin{aligned}
\mu_{[X_2]_{1/q}|X_1} &= [n-x_1]_q \frac{\theta_2}{1+\theta_2 q^{n-x_1-1}}, \\
(\sigma_{[X_2]_{1/q}|X_1})^2 &= \frac{1-q}{q}[n-x_1]_q^2 \frac{\theta_2^2}{(1+\theta_2 q^{n-x_1-1})^2(1+\theta_2 q^{n-x_1-2})} \\
&\quad + [n-x_1]_q \frac{\theta_2}{(1+\theta_2 q^{n-x_1-1})(1+\theta_2 q^{n-x_1-2})}
\end{aligned}$$

Note

Conditional q-mean, $\mu_{[X_2]_{1/q}|X_1}$: *q-regression curve*.

A Bivariate Stieltjes-Wigert Distribution as a Limiting Distribution

Theorem

Let $\theta_1 = \theta_{1,n} = q^{-\alpha_1 n}$ *and* $\theta_2 = \theta_{2,n} = q^{-\alpha_2 n}$ *with* $0 < a_1, a_2 < 1$ *constants and* $0 < q < 1$. *Then, for* $n \to \infty$, *the q-trinomial distribution of the first kind is approximated by a deformed standardized bivariate continuous Stieltjes-Wigert distribution as follows:*

$$f^B_{X_1,X_2}(x_1,x_2) \cong \frac{q^{-7/4}\log q^{-1}}{2\pi(q^{-1}-1)\sigma_{[X_1]_{1/q}}\sigma_{[X_2]_{1/q}|X_1}} q^{-(x_1+x_2)}$$
$$\cdot \left(q^{-3/2}(1-q)^{1/2}\frac{[x_1]_{1/q}-\mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_{1/q}}}+q^{-1}\right)^{-1/2}$$
$$\cdot \left(q^{-3/2}(1-q)^{1/2}\frac{[x_2]_q-\mu_{[X_2]_{1/q}|X_1}}{\sigma_{[X_2]_{1/q}|X_1}}+q^{-1}\right)^{-1/2}$$
$$\cdot \exp\left(\frac{1}{2\log q}\log^2\left(q^{-3/2}(1-q)^{1/2}\frac{[x_1]_{1/q}-\mu_{[X_1]_{1/q}}}{\sigma_{[X_1]_{1/q}}}+q^{-1}\right)\right)$$
$$\cdot \exp\left(\frac{1}{2\log q}\log^2\left(q^{-3/2}(1-q)^{1/2}\frac{[x_2]_q-\mu_{[X_2]_{1/q}|X_1}}{\sigma_{[X_2]_{1/q}|X_1}}+q^{-1}\right)\right),$$
$$, \qquad x_1,\ x_2 \geq 0.$$

Corollary

For $n \to \infty$ the following approximation holds

$$P\left(a_1 \le X_1 \le b_1, a_2 \le X_2 \le b_2\right) \cong \frac{1}{4}\left(\mathrm{Erf}(u_{b_1+1}) - \mathrm{Erf}(u_{a_1})\right) \cdot \left(\mathrm{Erf}(v_{b_2+1}) - \mathrm{Erf}(v_{a_2})\right), 0 \le a_i < b_i, i = 1, 2,$$

$$\begin{aligned} u_a &= \frac{1}{(2\log q^{-1})^{1/2}} \\ &\cdot \log\left(q^{-3/2}(1-q)^{1/2}\frac{[a-1/2]_{1/q} - \mu_{[X_1]_q}}{\sigma_{[X_1]_q}} + q^{-1}\right) - \frac{\sqrt{2\log q^{-1}}}{4}, \\ v_a &= \frac{1}{(2\log q^{-1})^{1/2}} \\ &\cdot \log\left(q^{-3/2}(1-q)^{1/2}\frac{[a-1/2]_{1/q} - \mu_{[X_2]_q|X_1}}{\sigma_{[X_2]_q|X_1}} + q^{-1}\right) - \frac{\sqrt{2\log q^{-1}}}{4}. \end{aligned}$$

Note

Numerical calculations (Mathematica, q-Series Package)-Indication of very good convergence even for moderate values of $n = 30, 50$

Asymptotic Behaviour of Multivariate Absorption Disitribution (M.V., 2020)

Marginal probability function of Y_k: Univariate Absorption

Marginal probability functions of Y_i, $i = 1, \ldots, k-1$, $k \geq 2$: Not Univariate Absorptions

q-means and q-variances of random variables Y_i, $i = 1, \ldots, k-1$, $k \geq 2$ cannot be found

Distributions of the conditional r.vs $Y_{k-1}|Y_k, Y_{k-2}|(Y_{k-1}, Y_k), \ldots, Y_1|(Y_2, \ldots, Y_k)$: Univariate Absorptions

Conditional q-means, q-variances:

$$\mu_{[Y_{k-1}]_q|Y_k}, \mu_{[Y_{k-2}]_q|(Y_{k-1},Y_k)}, \cdots, \mu_{[Y_1]_q|(Y_2,\ldots,Y_k)},$$

$$\sigma^2_{[Y_{k-1}]_q|Y_k}, \sigma^2_{[Y_{k-2}]_q|(Y_{k-1},Y_k)}, \cdots, \sigma^2_{[Y_1]_q|(Y_2,\ldots,Y_k)}$$

The mean and the variance of the deformed variable $[Y_k]_q$ are given by

$$\begin{aligned} \mu_{[Y_k]_q} &= E\left([Y_k]_q\right) = (1-q)[n]_q[m_k]_q \\ &\text{and} \\ (\sigma_{[Y_k]_q})^2 &= V\left([Y_k]_q\right) \\ &= q(1-q)^2[n]_{k,q}[m_k]_{2,q} \\ &\quad -(1-q)^2[n]_q^2[m_k]_q^2 + (1-q)[n]_q[m_k]_q, \end{aligned} \tag{2}$$

respectively.

The conditional mean and the conditional variance of the deformed variable $[Y_{k-1}]_q$ given $Y_k = y_k$ are given by

$$\begin{aligned} \mu_{[Y_{k-1}]_q|Y_k} &= E\left([Y_{k-1}]_q|y_k\right) = (1-q)[n-y_k]_q[m_{k-1}]_q \\ &\text{and} \\ (\sigma_{[Y_{k-1}]_q|Y_{n,k}})^2 &= V\left([Y_{k-1}]_q|y_k\right) \\ &= q(1-q)^2[n-y_k]_{2,q}[m_{k-1}]_{2,q} \\ &\quad -(1-q)^2[n-y_k]_q^2[m_{k-1}]_q^2 + (1-q)[n-y_k]_q[m_{k-1}]_q, \end{aligned}$$

respectively.

The conditional mean and conditional variance of the deformed variables $[Y_j]_q$ given $Y_{j+1} = y_{j+1}, \ldots, y_{n,k} = y_k$, $j = 1, \ldots, k-1$, $k \geq 2$, are given respectively by

$$\mu_{[Y_j]_q|\left(Y_{j+1},Y_{j+2},\ldots,Y_k\right)} = E\left([Y_j]_q \middle| \left(y_{j+1}, y_{j+2}, \ldots, Y_k\right)\right)$$

$$= (1-q)\left[n - \sum_{i=j+1}^{k} y_i\right]_q [m_j]_q,$$

$$\sigma^2_{[Y_j]_q|\left(Y_{j+1},Y_{j+2},\ldots,Y_k\right)} = V\left([Y_j]_q \middle| \left(y_{j+1}, y_{j+2}, \ldots, Y_k\right)\right)$$

$$= q(1-q)^2\left[n - \sum_{i=j+1}^{k} y_i\right]_{2,q} [m_j]_{2,q} - (1-q)^2\left[n - \sum_{i=j+1}^{k} y_i\right]_q^2 [m_j]_q^2$$

$$+(1-q)\left[n - \sum_{i=j+1}^{k} y_i\right]_q [m_j]_q.$$

Note

The conditional q-means, $\mu_{[Y_j]_q|(Y_{j+1},Y_{n,j+2},\ldots,Y_k)}$, $1 \leq j \leq k-2$, $k \geq 3$, can be interpreted as q-regression hyperplanes.

Asymptotic Behaviour of Multivariate Absorption Distribution

Theorem

Let $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $m_j = O(n), j = 1, 2, \ldots, k$. Then, for $n \to \infty$, the multivariate absorption distribution is approximated by a deformed multivariate standardized Gaussian distribution as follows:

$$f_{\mathcal{Y}}(y_1, y_2, \ldots, y_k) \cong \left(\frac{\log q^{-1}}{2\pi(1-q)}\right)^{k/2} \frac{q^{\sum_{j=1}^{k} y_j}}{\sigma_{[Y_k]_q} \prod_{j=2}^{k} \sigma_{[Y_{j-1}]_q|(Y_j,\ldots,Y_k)}}$$

$$\cdot \exp\left(\frac{1-q}{2\log q}\left(\left(\frac{[y_k]_q - \mu_{[Y_k]_q}}{\sigma_{[Y_k]_q}}\right)^2 + \sum_{j=2}^{k}\left(\frac{[y_{j-1}]_q - \mu_{[Y_{j-1}]_q|(Y_j,\ldots,Y_k)}}{\sigma_{[Y_{j-1}]_q|(Y_j,\ldots,Y_k)}}\right)^2\right)\right),$$

$y_j \geq 0, j = 1, 2, \ldots, k.$

Sketch Proof

- $Z_k = \frac{[Y_k]_q - \mu_{[Y_k]_q}}{\sigma_{[Y_k]_q}}$,
- $Z_j = \frac{[Y_j]_q - \mu_{[Y_j]_q \mid \left(Y_{j+1}, Y_{j+2}, \ldots, Y_k\right)}}{\sigma_{[Y_j]_q \mid \left(Y_{j+1}, Y_{nj+2}, \ldots, Y_k\right)}}, \; j = 1, \ldots, k-1, \; k \geq 2$
- q-Stirling type
- Pointwise convergence techniques applied to the joint probability function

Asymptotic Behaviour of Bivariate Absorption Distribution (M.V., 2024)

Let the discrete bivariate random variable (Y_1, Y_2) with joint probability function

$$f_{Y_1,Y_2}(y_1, y_2) = \binom{n}{y_1, y_2}_q (1-q)^{y_1+y_2} q^{(\nu-y_1-y_2)(n-y_1-y_2)} q^{y_1(\nu_2-y_2)} [\nu_1]_{y_1,q} [\nu_2]_{y_2,q},$$

where $y_j = 1, 2, \ldots, n$, $j = 1, 2$, with $y_1 + y_2 \leq n$, and $\nu = \nu_1 + \nu_2$, ν_1, ν_2 nonnegative integers.

Marginal probability function of Y_2: Univariate Absorption

The marginal probability function of the random variable Y_2, is distributed according to the univariate absorption distribution with probability function

$$f_{Y_2}(y_2) = \binom{n}{y_2}_q (1-q)^{y_2} q^{(\nu_2 - y_2)(n-y_2)} [\nu_2]_{y_2,q}, \quad y_2 = 0, 1, 2, \ldots, n,$$

for $0 < q < 1$ and $n \leq \nu_2$.

q-Mean, q-Variance

$$\mu_{[Y_2]_q} = E\left([Y_2]_q\right) = (1-q)[n]_q[\nu_2]_q$$

$$\begin{aligned}(\sigma_{[Y_2]_q})^2 &= V\left([Y_2]_q\right) \\ &= (1-q)^2[n]_{2,q}[\nu_2]_{2,q} - (1-q)^2[n]_q^2[\nu_2]_q^2 + (1-q)[n]_q[\nu_2]_q\end{aligned}$$

Marginal probability function of Y_1: Not Univariate Absorption

- q-mean and q variance of random variable Y_1 cannot be inferred from the corresponding q-moments of the univariate absorption distribution
- q-mean and q-variance cannot be found explicitly either directly or indirectly

Distribution of the conditional random variable $Y_1|Y_2$: Univariate Absorption

The conditional random variable $Y_1|Y_2$, is distributed according to the univariate absorption distribution with probability function

$$f_{Y_1|Y_2}(y_1|y_2) = \binom{n-y_2}{y_1}_q (1-q)^{y_1} q^{(\nu_1-y_1)(n-y_1-y_2)} [\nu_1]_{y_1,q},$$

$y_1 = 0, 1, 2, \ldots, n-y_2$, $0 < q < 1$, $n - y_2 \leq \nu_1$.

Conditional q-Mean, q-Variance

Conditional mean and conditional variance of the deformed variable $[Y_1]_q$ given $Y_2 = y_2$:

$$\mu_{[Y_1]_q|Y_2} = E\left([Y_1]_q|y_2\right) = (1-q)[n-y_2]_q[\nu_1]_q,$$

$$\begin{aligned}(\sigma_{[Y_1]_q|Y_2})^2 &= V\left([Y_1]_q|y_2\right)\\ &= (1-q)^2\left([n-y_2]_{2,q}[\nu_1]_{2,q} - [n-y_2]_q^2[\nu_1]_q^2\right)\\ &\quad +(1-q)[n-y_2]_q[\nu_2]_q.\end{aligned}$$

Note

Conditional q-Mean: q-Regression Curve

Asymptotic Behaviour of Bivariate Absorption Distribution (M.V., 2024)

Theorem

Let $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $\nu_i = O(n)$, $i = 1, 2$. Then, for $n \to \infty$, the bivariate absorption distribution is approximated by a deformed bivariate standardized Gaussian distribution as follows:

$$f_{Y_1,Y_2}(y_1, y_2) \cong \frac{\log q^{-1}}{2\pi(1-q)\sigma_{[Y_2]_q}\sigma_{[Y_1]_q|Y_2}} q^{y_1+y_2}$$
$$\cdot \exp\left(-\frac{1-q}{2\log q^{-1}} \cdot \left(\left(\frac{[y_2]_q - \mu_{[Y_2]_q}}{\sigma_{[Y_2]_q}}\right)^2 + \left(\frac{[y_1]_q - \mu_{[Y_1]_q|Y_2}}{\sigma_{[Y_1]_q|Y_2}}\right)^2\right)\right),$$
$$y_1, y_2 \geq 0.$$

Asymptotic Behaviour of Multivariate q-Hypergeometric Disitribution

Marginal probability function of Y_1: Univariate q-Hypergeometric

Marginal probability functions of W_i, $i = 2, \ldots, k$, $k \geq 2$: Not q-Hypergeometrics

q-means and q-variances of random variables W_i, $i = 1, \ldots, k-1$, $k \geq 2$ cannot be found explicitly

Distributions of the conditional r.vs $W_2|W_1, W_3|(W_1, W_2), \ldots, W_k|(W_1, \ldots, W_{k-1})$: Univariate q-Hypergeometric

Conditional q-means, q-variances:

$$\mu_{[W_2]_q|W_1}, \mu_{[W_3]_q|(W_1,W_2)}, \cdots, \mu_{[W_k]_q|(W_1,\ldots,W_{k-1})},$$

$$\sigma^2_{[W_2]_q|W_1}, \sigma^2_{[W_3]_q|(W_1,W_2)}, \cdots, \sigma^2_{[W_k]_q|(W_1,\ldots,W_{k-1})}$$

The mean and the variance of the deformed variable $[W_1]_q$ are given by

$$\mu_{[W_1]_q} = E([W_1]_q) = \frac{[n]_q[\nu_1]_q}{[\nu]_q}$$

and

$$(\sigma_{[W_1]_q})^2 = V([W_k]_q) = q\frac{[n]_{2,q}[\nu_1]_{2,q}}{[\nu]_{2,q}} + \frac{[n]_q[\nu_1]_q}{[\nu]_q} - \left(\frac{[n]_q[\nu_1]_q}{[\nu]_q}\right)^2.$$

respectively.

The conditional mean and the conditional variance of the deformed variable $[W_2]_q$ given $W_1 = w_1$ are given by

$$\mu_{[W_2]_q|W_1} = E\left([Y_{n,2}]_q|w_1\right) = \frac{[n-w_1]_q[\nu_2]_q}{[\nu-\nu_1]_q}$$

and

$$\left(\sigma_{[W_2]_q|W_1}\right)^2 = V\left([W_2]_q|w_1\right) = q\frac{[n-w_1]_{2,q}[\nu_2]_{2,q}}{[\nu-\nu_1]_{2,q}} + \frac{[n-w_1]_q[\nu_2]_q}{[\nu-\nu_1]_q} - \left(\frac{[n-w_1]_q[\nu_2]_q}{[\nu-\nu_1]_q}\right)^2$$

respectively.

The conditional mean and conditional variance of the deformed variables $[W_j]_q$ given $W_1 = w_1, \ldots, W_{j-1} = w_{j-1}$, $j = 2, \ldots, k$, $k \geq 2$, are given respectively by

$$\mu_{[W_j]_q|\left(W_1,W_2,\ldots,W_{j-1}\right)} = E\left([W_j]_q|\left(w_1, w_2, \ldots, w_{j-1}\right)\right) = \frac{\left[n - \sum_{i=1}^{j-1} w_i\right]_q [\nu_j]_q}{[\nu - \sum_{i=1}^{j-1} \nu_i]_q},$$

$$\sigma^2_{[W_j]_q|\left(W_1,W_2,\ldots,W_{j-1}\right)} = V\left([W_j]_q|\left(w_1, w_2, \ldots, w_{j-1}\right)\right) = \frac{q[n - \sum_{i=1}^{j-1} w_i]_{2,q}[\nu_j]_{2,q}}{[\nu - \sum_{i=1}^{j-1} \nu_i]_{2,q}} + \frac{\left[n - \sum_{i=1}^{j-1} w_i\right]_q [\nu_j]_q}{[\nu - \sum_{i=1}^{j-1} \nu_i]_q} - \left(\frac{\left[n - \sum_{i=1}^{j-1} w_i\right]_q [\nu_j]_q}{[\nu - \sum_{i=1}^{j-1} \nu_i]_q}\right)^2.$$

Note

The conditional q-means, $\mu_{[W_j]_q|(W_1,W_2,\ldots,W_{j-1})}$, $2 \leq j \leq k$, $k \geq 3$, can be interpreted as q-regression hyperplanes.

Theorem

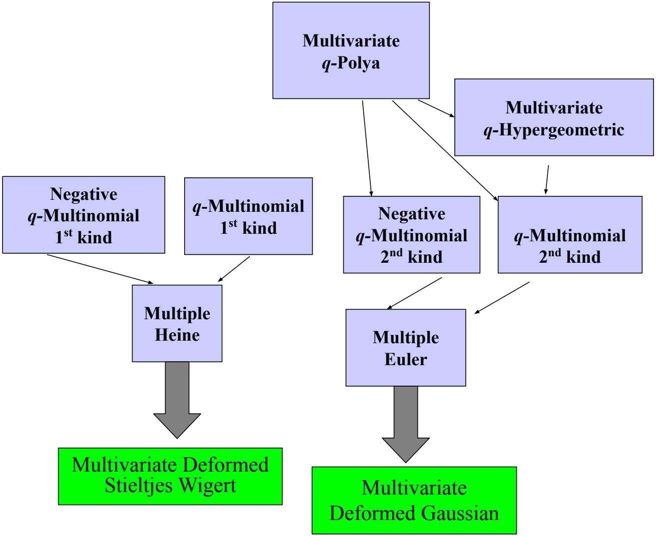
Let $q = q(n)$ with $q(n) \to 1$, as $n \to \infty$, $q(n)^n = \Omega(1)$ and $\nu = O(n)$. Then, for $n \to \infty$, the multivariate q-Hypergeometric distribution is approximated by a deformed multivariate standardized Gaussian distribution as follows:

$$f_{\mathcal{W}}(w_1, w_2, \ldots, w_k) \cong \left(\frac{\log q^{-1}}{2\pi(q^{-1}-1)}\right)^{k/2} \frac{q^{\sum_{i=1}^{k} w_i}}{\sigma_{[W_1]_q} \prod_{j=2}^{k} \sigma_{[W_j]_q|(W_1,W_2,\ldots,W_{j-1})}}$$

$$\cdot \exp\left(\frac{1-q}{2\log q}\left(\left(\frac{[w_1]_q - \mu_{[W_1]_q}}{\sigma_{[W_1]_q}}\right)^2 + \sum_{j=2}^{k}\left(\frac{[w_j]_q - \mu_{[W_j]_q|(W_1,\ldots,W_{j-1})}}{\sigma_{[W_j]_q|(W_1,\ldots,W_{j-1})}}\right)^2\right)\right),$$

$$w_j \geq 0, j = 1, 2, \ldots, k, k \geq 2.$$

Asymptotic Behaviour of Multivariate Discrete q-Distributions



Discussion and Further Study

- Is it possible to unify the study of the asymptotic behavior of univariate and multivariate discrete q-distributions without proving the local limit theorems separately for each distribution?
- Charalambides: "Malvina, once I finish my new book on Multivariate Discrete q- Distributions, I would like to discuss the unification of local limit theorems"

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Thank you!!!