Normal Deviation of Gamma Processes in Random Environment with Applications

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Abstract

We consider Gamma processes of homogeneous type which live in a random environment or media represented by a pure jump Markov process. The aim of this paper is to approximate such gamma processes by a diffusion. Since gamma processes are increasing, the diffusion approximation requires an average approximation first. This averaged process will serve as an equilibrium to the initial gamma process. We present two main results: averaging and normal deviation. An application for degradation systems in reliability modeling is discussed.[2]

Keywords: Gamma process, random environment, normal deviation, averaging, Markov process.

1 Introduction

Stochastic modeling problems in random environment or media are of great importance as they provide more realistic representation for real systems. This kind of models is also more advanced from a mathematical point of view. Degradation phenomena are generally slow while environmental changes are fast. In this situation stochastic approximations can be considered in order to simplify the complex systems and provide simpler methods to handle them.

In reliability theory or survival analysis the lifetime T of an item is defined by the time at which its degradation path reaches a critical level, a threshold value, (see, e.g., [14, 17, 21, 23, 24]). When the item lives into a fixed environment the evaluation of reliability, $\mathbb{P}(T > t)$ is easy. But when the item lives into a random environment, then the evaluation of reliability and of other performance indices, is more difficult. The latter is the type of problems that we are interested in.

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A gamma process is a particular case of a Lévy process, see, e.g., [2, 22] and, in particular, it is a Markov process, as independent and positive increments jump process. It is a good candidate for monotone degradation phenomena. It is, in fact, extensively used in many applications, where degradation of systems is observed, such as survival analysis [17, 24], reliability [1, 21, 23], damage accumulation models, crack propagation in fracture mechanics, [17], and risk theory [9], etc.

There are homogeneous, non-homogeneous and some extended generalizations of gamma processes, see [9, 5, 4, 8]. We consider a homogeneous gamma process in a random environment and propose an averaging and a normal deviation results. The random environment is assumed to be a Markov process with general state space. When the environment is fixed, the considered gamma process is a homogeneous.

In this paper we obtain an averaging result for the time-scaled gamma process as well as a normal deviation, or diffusion approximation with equilibrium, for the time-scaled gamma process centered by the averaged process. In fact, as the gamma process is nondecreasing it does not satisfy the usual balance condition and can not provide a diffusion approximation in a usual way. We consider the weighted difference of the time-rescaled gamma process from the averaged process. This new process converges weakly in the Skorohod space to a Wiener process; see, e.g., [10, 13, 14, 16].

The next section provides some preliminaries on gamma processes and Markov processes. Section 3 provides the main results; Section 4 provides the proofs; Section 5 provides an application in reliability; and finally Section 6 provides some concluding remarks.

2 Preliminaries

In this section we give definitions of gamma and Markov processes which will be needed in later sections. The gamma process may be defined either as a particular case of Lévy process (a subordinator), or directly as an independent increment process. The first definition provides us a more insightful information for its probabilistic structure and its trajectory properties which we point out briefly while the second definition is more operational. We also provide a counter-example, a process with marginal gamma distribution which is not a gamma process.

Gamma processes. Let us consider a Poisson random measure $\mu(du, ds)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with mean measure Π defined by

$$\Pi(du, ds) = bu^{-c-1}e^{-\alpha u}duds$$

where $\alpha > 0$, $b > 0$ and $c < 1$, are parameters. Define the stochastic process $Z(t), t \geq 0$, as

$$Z(t) := \int_{\mathbb{R}_+ \times (0,t]} u\mu(du, ds), \quad t > 0, \quad Z(0) = 0; \tag{1}$$

which is an increasing process with Laplace transform

$$\mathbf{E}[e^{-\lambda Z(t)}] = \exp[-t \int_{(0,\infty)} (1 - e^{-\lambda u})\nu(du)] \tag{2}$$

where $\nu(du)ds = \Pi(du, ds)$.

If we take $c = 0$, then the process $Z(t), t \geq 0$, (1), is said to be a gamma process. This process is a Lévy process with Lévy measure

$$\nu(du) = bu^{-1}e^{-\alpha u}du, \qquad u > 0,$$

which satisfies the following integrability condition

$$\int_{(0,\infty)} (1 \wedge x)\nu(dx) < \infty. \tag{3}$$

It is worth noting that this integral is bounded by $1/\alpha$. So, the measure ν is σ-finite. Such gamma process $Z(t)$ increases by jumps only, and, as $\nu(0, \infty) = +\infty$, the set of mass points $\{s_i\}$ is dense in $\mathbb{R}_+$, and has infinitely many jumps in every open interval (s, t), $s < t$.

We can also define the gamma process directly as follows. A process $(Z(t), t \geq 0)$ is said to be a homogeneous gamma process, denoted by $GP(bt, \alpha)$, if the following conditions are fulfilled:

1. $Z(0) = 0$;

2. Z has independent increments;

3. The increments $Z(t + h) - Z(t)$, for $t \geq 0, h > 0$, are stationary and follow the gamma distribution, i.e., its pdf f_h is the $Ga(bh, \alpha)$, see (4) below.

The name of the gamma process comes from the fact that the pdf of its marginal distribution is the gamma distribution, $Ga(bt, \alpha)$, i.e., its density function is

$$f_t(x) = \frac{(\alpha x)^{bt-1}}{\Gamma(bt)} \alpha e^{-\alpha x}, \quad x \geq 0, \tag{4}$$

where bt is the shape parameter and $\alpha > 0$ the scale or rate parameter; and $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, is the gamma Eulerian function defined for $x > 0$. Its mean is bt/α and its variance is bt/α^2. The Laplace transform of $Z(t)$ is

$$\mathbf{E}[e^{-\lambda Z(t)}] = (\frac{\alpha}{\lambda + \alpha})^{bt} = \exp\Big[-t \int_{(0,\infty)} (1 - e^{-\lambda x}) \nu(dx) \Big]$$

for $\lambda \geq 0$, with

$$\nu(dx) = \frac{b e^{-\alpha x}}{x} dx, \quad x > 0,$$

its Lévy measure.

Let us give an example of a process where the marginal distributions are gamma, but it is not a gamma process.

Counter-example. The Cox-Ingersoll-Ross process [7], is defined by the stochastic differential equation

$$dX(t) = \lambda(\rho - X(t))dt + \sigma \sqrt{|X(t)|} dW(t)$$

where λ, ρ, σ, are positive constants and $W(t)$ is a standard Wiener process. This process has the gamma distribution, $Ga(2\lambda\rho/\sigma^2; 2\lambda/\sigma^2)$ as stationary probability. Of course, when its initial law is this gamma distribution then its law at any time $t > 0$ is gamma, but, obviously, this process is not a gamma process.

Markov process. Let us consider a regular pure jump Markov process X, with state space the measurable space $(E, \mathcal{E})$, with countably generated σ-algebra, (see, e.g., [18]). We suppose also that X is cadlag, i.e., continuous on the right having left limits at any point of time $t > 0$; see, e.g., [12, 14]. Denote by $0 = S_0 < S_1 < ...$ the jump times and $J_n, n \geq 0$, the successive visited states. As usually, we denote by $\mathbf{P}_x(\cdot)$ the conditional probability $\mathbf{P}(\cdot \mid J_0 = x)$ and by $\mathbf{E}_x$ the corresponding expectation operator. Define also the counting process $N(t) := \inf\{n > 0 : S_n \leq t\}$, which counts the number of jumps into the time interval $(0, t]$.

The Markov process, $X(t), t \geq 0$, is defined by its generator, A,

$$A\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)], \tag{5}$$

where $q(x), x \in E$, is the intensity function of jumps, and P is the transition kernel of the embedded Markov chain (J_n) of the Markov process X. We consider that the Markov process X is uniformly ergodic, with ergodic probability π, i.e., $\pi A\varphi = 0$. Define also the transition probability $P_t(x, B) := \mathbb{P}(X(t) \in B \mid X(0) = x)$ and the transition operator P_t, $t \geq 0$ of X.

Define now the stationary projector Π and the potential operator R_0 of the above Markov process, X, as follows. Let Π be the stationary projector operator defined by

$$\Pi\varphi(x) = \int_E \pi(dv)\varphi(v)\mathbf{1}_E(x).$$

The potential operator R_0 of P_t is defined by

$$R_0 = \int_0^\infty (P_t - \Pi)dt$$

and

$$AR_0 = R_0 A = \Pi - I.$$

3 Results

3.1 Gamma process in random environment

Let us consider a right continuous Markov process, say $X(t), t \geq 0$, with state space $(E, \mathcal{E})$, a compact Borel measurable space, which represent the random environment.

Consider also a gamma process, say $Z(t), t \geq 0$, i.e., $GP(bt, \alpha)$, defined by its Laplace transform

$$\mathbf{E}[e^{-\lambda Z(t)}] = \exp\Big[- t \int_{(0,\infty)} (1 - e^{-\lambda u}) b u^{-1} e^{-\alpha u} du \Big]$$

where $\nu(du) = bu^{-1}e^{-\alpha u}du$ is the Lévy measure which satisfy the above condition (3).

For this gamma process, the shape and rate parameters, at time $t \geq 0$, are now functions of the states $x \in E$, i.e., $b = b(x)$ and $\alpha = \alpha(x)$, respectively, on the event $\{X(t) = x\}$. Consequently, the Lévy measure, ν, depends also on the state $x \in E$, i.e.,

$$\nu = \nu(x, du) = b(x)u^{-1}e^{-\alpha(x)u}du.$$

Define also the natural filtration $\mathcal{F}_t := \sigma(X(s), 0 \leq s \leq t)$, for $t \geq 0$.

The following assumptions are important in the sequel.

A1: We suppose here that for any fixed $x \in E$, the gamma process $Z(t)$, $GP(b(x)t, \alpha(x))$, and the process $X(t)$ are independent.

A2: We suppose that $\kappa := \inf\{\alpha(x) : x \in E\} > 0$, and moreover

$$\int_E \pi(dx) b(x) < \infty, \quad \text{for any} \quad t > 0.$$

This implies that the mean value and the variance of $Z(t)$ are finite, for any finite t.

Proposition 3.1. *Assume that conditions A1-A2 are satisfied. Then we have*

$$\mathbf{E}_x[e^{-\lambda Z(t)} \mid \mathcal{F}_t] = \exp\Big[-\int_0^t b(X(s)) \ln\Big(1 + \frac{\lambda}{\alpha(X(s))}\Big) ds\Big] \tag{6}$$

for $0 \leq \lambda \leq \kappa$.

From Equation (6), it is not difficult to show that

$$\mathbf{E}_x[e^{-\lambda Z(t)} \mid \mathcal{F}_t] = \exp\Big[-\int_0^t b(X(s)) ds \int_{(0,\infty)} \frac{1 - e^{-\lambda u}}{u} e^{-\alpha(X(s))u} du\Big]. \tag{7}$$

3.2 Average approximation

Let us consider now the above formulation in a series scheme, for $\varepsilon > 0$, the series parameter as follows. Define the Markov processes in rescaled time $X^\varepsilon(t) = X(t/\varepsilon)$, and their natural filtrations $\mathcal{F}_t^\varepsilon := \sigma(X^\varepsilon(s), 0 \leq s \leq t)$, for $t \geq 0$, $\varepsilon > 0$. Now define the family of processes

$$Y^\varepsilon(t) := Y^\varepsilon(t, \lambda, X) := \mathbf{E}[e^{-\lambda Z^\varepsilon(t)} \mid \mathcal{F}_t^\varepsilon] \tag{8}$$

and the function $a(x; \lambda) := b(x) \ln(1 + \frac{\lambda}{\alpha(x)})$, $x \in E$, $\lambda > 0$.

Let us denote by $\Rightarrow$ the weak convergence in the Skorohod space of cadlag functions, see, e.g. [3, 10, 13, 14].

Theorem 3.1. *Assume that conditions A1-A2 are satisfied. Then the following weak convergence holds*

$$Y^\varepsilon(t) \Rightarrow \exp(-\hat{a}(\lambda)t), \quad \textit{as} \quad \varepsilon \downarrow 0,$$

where $\hat{a}(\lambda) := \int_E \pi(dx) \alpha(x; \lambda)$*, and the limit is the Laplace transform of an independent increment process.*

Moreover, we have, for any fixed $T > 0$,

$$\sup_{t \le T} |Y^\varepsilon(t) - \exp(-\hat{a}(\lambda)t)| \xrightarrow{P} 0, \quad \text{as} \quad \varepsilon \downarrow 0.$$

We can indeed prove that $e^{-\hat{a}(\lambda)t}$, is the Laplace transform of a positive r.v., see Lemma 4.1, in Section 4.

The averaging principle considered here is also known in the literature as the stochastic averaging Bogolyubov principle; see, e.g., [20].

3.3 Normal deviation

Let us define the family of stochastic processes

$$V^\varepsilon(t) := \exp\Big[- \int_0^t a(X(s/\varepsilon^2); \lambda) ds \Big], \quad t \ge 0, \quad \varepsilon > 0. \tag{9}$$

In the usual diffusion approximation scheme we have to consider an additional condition, the balance condition, see, e.g., [14], which in our setting is written as $\hat{a}(\lambda) = 0$. But now we have $\hat{a}(\lambda) > 0$. In this case, we can obtain a diffusion approximation by equilibrate the above process by the averaged one. So, we have a diffusion approximation with equilibrium or a normal deviation theorem as follows.

Let us consider the processes

$$S^\varepsilon(t) := \int_0^t a(X(s/\varepsilon^2); \lambda) ds.$$

Then we obtain the following result.

Theorem 3.2. *Assume that conditions A1-A2 are satisfied. Then the processes* $\varepsilon^{-1}(S^\varepsilon(t) - \hat{a}(\lambda)t), t \ge 0, \varepsilon > 0$, *converge weakly, as* $\varepsilon \downarrow 0$, *to the process* $\sigma(\lambda)W(t)$, *where* $W(t), t \ge 0$, *is a standard Wiener process, and*

$$\sigma^2(\lambda) := 2 \int_E \pi(dx)[(a(x,\lambda) - \hat{a}(\lambda))(R_0 - I)(a(x,\lambda) - \hat{a}(\lambda)) + (a(x,\lambda) - \hat{a}(\lambda))^2].$$

Moreover, this result can also be written as follows

$$\varepsilon^{-1}(-\ln V^\varepsilon(t) - \hat{a}(\lambda)t) \Rightarrow \sigma(\lambda)W(t), \qquad \varepsilon \downarrow 0.$$

4 Proofs

PROOF of PROPOSITION 3.1. Based on the fact that the gamma process Z is of independent increments, and writing $Z(t) = \sum_{n=1}^{N(t)}(Z(S_n) - Z(S_{n-1})) + (Z(t) - Z(S_{N(t)}))$, together with assumption A1, we can write:

$$\begin{aligned}
\mathbf{E}_x[e^{-\lambda Z(t)} \mid \mathcal{F}_t] &= \prod_{k=1}^{N(t)} \mathbf{E}_x[e^{-\lambda(Z(S_n)-Z(S_{n-1}))} \mid \mathcal{F}_t] \times \mathbf{E}_x[e^{-\lambda(Z(t)-Z(S_{N(t)}))} \mid \mathcal{F}_t] \\
&= \prod_{k=1}^{N(t)} \Big(\frac{\alpha(J_{n-1})}{\lambda + \alpha(J_{n-1})}\Big)^{b(J_{n-1})(S_n - S_{n-1})} \times \Big(\frac{\alpha(J_{N(t)})}{\lambda + \alpha(J_{N(t)})}\Big)^{b(J_{N(t)})(t - S_{N(t)})} \\
&= \exp\Big\{ - \sum_{n=1}^{N(t)} b(J_{n-1})(S_n - S_{n-1}) \ln(1 + \frac{\lambda}{\alpha(J_{n-1})}) \\
&\quad - b(J_{N(t)})(t - S_{N(t)}) \ln(1 + \frac{\lambda}{\alpha(J_{N(t)})})\Big\} \\
&= \exp\Big\{ - \int_0^t b(X(s)) \ln(1 + \frac{\lambda}{\alpha(X(s))}) ds\Big\}.
\end{aligned}$$

The proof is completed. ∎

PROOF of THEOREM 3.1. The process $Y^\varepsilon(t)$ can be written as follows

$$Y^\varepsilon(t) = \exp\Big[- \xi^\varepsilon(t)\Big]$$

and

$$\xi^\varepsilon(t) := \int_0^t b(X^\varepsilon(s)) \ln(1 + \frac{\lambda}{\alpha(X^\varepsilon(s))}) ds$$

where the exponent is an integral functional of the Markov process (X_t). The generator of the coupled family of Markov processes $(\xi^\varepsilon(t), X^\varepsilon(t))$ is

$$L^\varepsilon = \varepsilon^{-1} A + D$$

where the operator A is defined by the above relation (5), and the operator D is defined by

$$D(x)\varphi(u) = a(x;\lambda)\varphi'(u).$$

Now following Proposition 5.1., in [14], by the solution of the singular perturbation problem, on the test functions $\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x)$, i.e.,

$$(\varepsilon^{-1}A + D)(\varphi + \varepsilon\varphi_1) = \hat{D}\varphi + \varepsilon\theta^\varepsilon$$

where θ^ε is a negligible operator, we obtain the limit generator $\hat{D}$, of the process ξ, by

$$\hat{D} = \Pi a(x;\lambda)\Pi.$$

Then the following weak convergence holds:

$$\xi^\varepsilon(t) \Rightarrow \hat{a}(\lambda)t, \quad \text{as} \quad \varepsilon \downarrow 0,$$

and the result follows from the continuous mapping theorem (see, e.g., [3]).

The above limit, i.e., $\exp(-\hat{a}(\lambda)t)$, is a completely monotone function [11] for every $t > 0$. Consequently, it is the Laplace transform of a stochastic process. Moreover, we have that $\hat{a}(\lambda) \to 0$, as $\lambda \to 0$, which means that the limit distribution is not defective. Now, we conclude by the continuity theorem of Laplace transforms. The proof that the limit process is of independent increments is given in the following lemma. ∎

Lemma 4.1. *The function* $\exp(-\hat{a}(\lambda)t)$*, for any* $t > 0$*, is the Laplace transform of a nonnegative infinitely divisible r.v. Hence, the process is of independent increments.*

Proof Let us define the function

$$\psi(\lambda) := \frac{da}{d\lambda}(x;\lambda) = \frac{b(x)}{\lambda + \alpha(x)}.$$

The $n-$th derivative of ψ is

$$\psi^{(n)}(\lambda) = (-1)^n n! \frac{b(x)}{(\lambda + \alpha(x))^{n+1}}.$$

Hence ψ is a completely monotone function, and then $\frac{da}{d\lambda}(x;\lambda)$ is a completely monotone function which implies that $\hat{a}(\lambda)$ is completely monotone too. Now, from Feller's criterion 2, p. 441 [11], we obtain that $\exp(-\hat{a}(\lambda))$ is the Laplace transform of a r.v. So is the function $\exp(-\hat{a}(\lambda)t)$, for any $t > 0$.

In order to prove the infinite divisibility, we consider the function $\hat{a}(\lambda)$, in the following form, see (7), which can be written as

$$\hat{a}(\lambda)t = \int_0^\infty \frac{1 - e^{-\lambda u}}{u} P(du),$$

where

$$P(du) := t \int_E \pi(dx) b(x) \exp(-\hat{a}(\lambda)u) du$$

From Theorem 2, p. 450, [11], we have to prove that the measure $P(du)$ satisfy

$$\int_1^\infty u^{-1}P(du) < \infty,$$

Indeed, we have:

$$\int_1^\infty \int_E \pi(dx)tb(x)u^{-1}e^{-\hat{\alpha}(x)u}du \leq \int_E \pi(dx)\frac{tb(x)}{\kappa} < \infty.$$

The last inequality is satisfied thanks to assumption A2. So, the Lemma is proved. ∎

Proof of Theorem 3.2. The result stated in this theorem is the diffusion approximation with equilibrium of an integral functional of a Markov process, see, e.g., for this type of theorems [14]. Let us write

$$\varepsilon^{-1}(S^\varepsilon(t) - \hat{a}(\lambda)t) = \varepsilon^{-1}\int_0^t [a(X(s/\varepsilon^2);\lambda) - \hat{a}(\lambda)]ds$$

For the function $C(x;\lambda) := a(x;\lambda) - \hat{a}(\lambda)$, the balance condition is obviously satisfied, i.e., $\Pi C = 0$. Let us consider the coupled Markov processes, $S^\varepsilon(t)$, $X(s/\varepsilon^2)$, $t \geq 0$, $\varepsilon > 0$. The generator of this processes is:

$$L^\varepsilon \varphi(u,x) := [\varepsilon^{-2}A + \varepsilon^{-1}B]\varphi(u,x),$$

where A is the generator of the Markov process X and B is the generator of the random evolution defined by $B(x;\lambda)\varphi(u) = a(x;\lambda)\varphi'(u)$. Now, by the solution of the following singular perturbation problem, on the test functions $\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)$,

$$L^\varepsilon \varphi^\varepsilon(u,x) = L\varphi(u) + \varepsilon\theta^\varepsilon(u,x),$$

where $\theta^\varepsilon(u,x)$ is a negligible operator, we obtain the limit operator L,

$$L\varphi(u) = \Pi[B(x;\lambda) - \hat{a}(\lambda)]R_0[B(x;\lambda) - \hat{a}(\lambda)]\Pi\varphi(u)$$

from which we obtain directly

$$L\varphi(u) = \frac{1}{2}\sigma^2(\lambda)\varphi''(u).$$

So, the limit process is $\sigma(\lambda)W(t)$, where $W(t)$ is a standard Wiener process, and the proof is completed. ∎

Remark 4.1. *From the chain rule, see, e.g., Theorem 20.9 in [25], we can conclude that the functional* $\exp\circ(-S^\varepsilon(t))$ *is Hadamard differentiable. So, from this and Theorem 3.2, we conclude that* $\varepsilon^{-1}(V^\varepsilon(t) - \exp(-\hat{a}(\lambda)t))$ *weakly converges to* $\hat{a}(\lambda)\exp(-\hat{a}(\lambda)W(t))$.

5 Toward applications

The above results can be applied in any domain where we have a model via a gamma process in a randomly varying environment. Let us give an example in reliability. When we have the degradation of an item (component, system, ...), modeled by a gamma process in a fixed environment, then we have an easy formulation of reliability as follows. Let us consider a constant threshold, say $a > 0$, a limit for item performance. In this case, we define the item lifetime, T, by $T(\omega) := \inf\{t > 0 : Z(t;\omega) \geq a\}$, see Figure 1. The reliability of this item is $R(t) = \mathbb{P}(T > t) = \mathbb{P}(Z(t) < a)$, and then for a gamma process, $GP(bt, \alpha)$, its expression is given as follows, see, e.g., [21, 23, 24],

$$R(t) = 1 - \frac{\Gamma(bt, a\alpha)}{\Gamma(bt)}, \quad t \geq 0$$

where $\Gamma(bt, x) = \int_x^\infty t^{bt-1}e^{-t}dt$, is the gamma incomplete function.

But when the same item lives in a random environment, then the situation is much more difficult and no explicit simple formulation exists to obtain the reliability function. In that case, the results presented in the previous sections are useful.

For example, let the evolution of degradation be described by a gamma process $Z(t)$, and the random environment by a Markov process $X(t)$, with two states, say 1(dry) and 2 (wet), and the generating matrix A

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

For this two state process, the stationary probability is $\pi_1 = \pi_2 = 1/2$. For the gamma process the parameters in state $i = 1, 2$ are denoted by b_i, α_i.

Hence the Laplace transform of the limit process, from Theorem 3.1, is

$$\exp[-\frac{1}{2}(b_1 \ln(1 + \frac{\lambda}{\alpha_1}) + b_2 \ln(1 + \frac{\lambda}{\alpha_2}))t].$$

By inversion of the Laplace transform, we obtain the limit process whose marginal law is given by the convolution of two gamma laws, i.e. $Ga(b_1t/2, \alpha_1) * Ga(b_2t/2, \alpha_2)$, or, equivalently, the limit process is the sum of two independent gamma processes, and the reliability can be calculated easily as previously in the fixed environment. In the case where the parameter α is not a function of the environment, i.e., $\alpha_1 = \alpha_2 = \alpha$, the limit process is the gamma $GP((b_1 + b_2)t/2, \alpha)$.

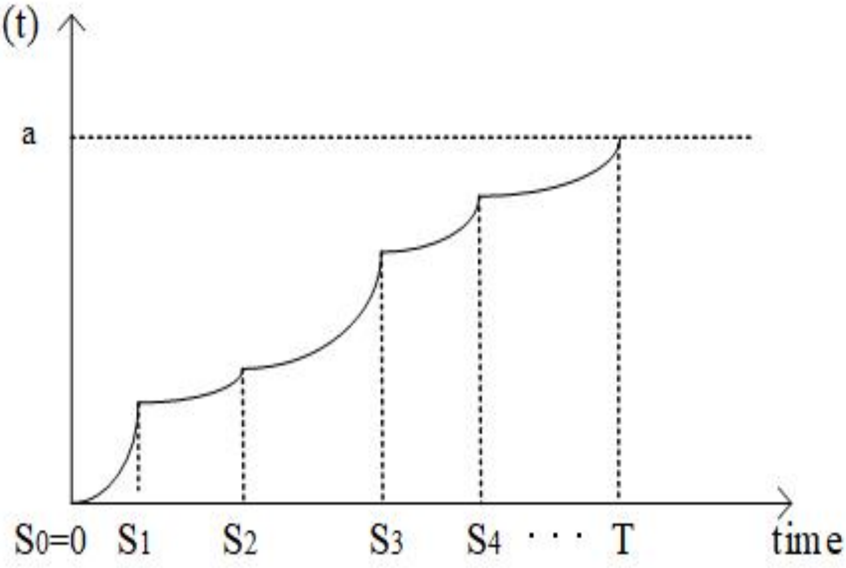

Figure 1: A gamma process in random environment, with threshold a and lifetime T. The times $S_1, S_2, ...$ are the jump times of the process X.

Another application may concern the statistics where we consider time-dependent covariates, see, e.g., [6, 17]. The time dependent covariates can be considered as a random environment. The results presented here may help to obtain, for example, asymptotic properties of estimators and also to approximate the survival function.

Result of Theorem 3.2 can also be used to obtain confidence intervals of the gamma process from the averaging process of Theorem 3.1.

6 Concluding remarks

For instance, the results presented here are rather theoretical but a more detailed study can be done towards applications, especially in reliability and survival analysis where gamma processes are used a lot. Similar results can be obtained for more general processes, for example, using semi-Markov processes as random environment; see, e.g., [18] .

The use of ergodic random environment is somewhat natural, but non-ergodic environment can also be considered; see [14, 16]. Reduced random environment can also be of interest to consider in order to simplify models, as in many cases it is important to keep in the model only essential characteristics and values of the environment. The discrete-time models can be used certainly in order to obtain direct computational results; see, e.g.,

[19].

Of course, inverting the Laplace transform of $\tilde{F}(\lambda) := \exp(-\hat{a}(\lambda)t)$ can be a challenging problem. To this end, we can use the inversion formula (see, e.g., [11])

$$F(x) := \lim_{a\to\infty} \sum_{n\leq\lambda x} \frac{(-a)^n}{n!}\tilde{F}^{(n)}(\lambda),$$

for any continuity point x of F.

For example, if $\hat{\alpha}(\lambda) = K\lambda$, $(K > 0)$, we obtain

$$F(x) = \begin{cases} 0 & \text{if} \quad x < Kt \\ 1 & \text{if} \quad x > Kt. \end{cases}$$

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